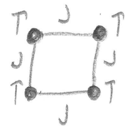
Heisenbergov model:

$$H = J\sum_{i=1}^{4} \vec{S}_i \cdot \vec{S}_{i+1}$$



operator celotnega spina $\vec{S} = \sum_{i=1}^{4} \vec{S}_i$ komutira s Hamiltonianom:

$$[H, \vec{S}]_\alpha = [H, S_\alpha] = \Big[J\sum_{i\beta} S_{i\beta} S_{i+1\beta}, \sum_j S_{j\alpha}\Big] = J\sum_{ij\beta} [S_{i\beta} S_{i+1\beta}, S_{j\alpha}] =$$

$\alpha = x, y, z$

$$= J\sum_{ij\beta}\Big([S_{i\beta}, S_{j\alpha}] S_{i+1\beta} + S_{i\beta}[S_{i+1\beta}, S_{j\alpha}]\Big) =$$

$$= J\sum_{ij\beta\gamma}\Big(i\hbar\delta_{ij}\varepsilon_{\beta\alpha\gamma} S_{i\gamma} S_{i+1\beta} + S_{i\beta}\delta_{i+1j}\, i\hbar\, \varepsilon_{\beta\alpha\gamma} S_{i+1\gamma}\Big) =$$

$$= Ji\hbar\sum_{i\beta\gamma}\varepsilon_{\beta\alpha\gamma}\big(S_{i\gamma} S_{i+1\beta} + S_{i\beta} S_{i+1\gamma}\big) =$$

$$= Ji\hbar\sum_{i\beta\gamma}\big(\varepsilon_{\beta\alpha\gamma} S_{i\gamma} S_{i+1\beta} + \varepsilon_{\gamma\alpha\beta} S_{i\gamma} S_{i+1\beta}\big) =$$

$$= Ji\hbar\sum_{i\beta\gamma}\underbrace{(\varepsilon_{\beta\alpha\gamma} + \varepsilon_{\gamma\alpha\beta})}_{=0} S_{i\gamma} S_{i+1\beta} = \phi$$

$\boxed{H, S_z, S^2 \text{ in } T \text{ komutirajo!}}$ Torej lahko najdem lastne funkcije, ki so hkrati lastn[e] funkcije vseh štirih operatorjev.

$$\hat{S}_z|\psi) = S_z|\psi) \qquad (\hbar \equiv 1)$$

$$S^2|\psi) = S(S+1)|\psi)$$

$$T|\psi) = e^{ika}|\psi) \quad k = -\tfrac{\pi}{2}, 0, \tfrac{\pi}{2}, \pi \quad \text{(zaradi periodičnih robnih pogojev)} \quad (a \equiv 1)$$

$$e^{ik} = -i, 1, i, -1$$

lastne funkcije torej lahko opišem s kvantnimi števili S_z, S in k!
Matrični elementi med stanji, ki se razlikujejo v vsaj enem od teh kvantnih števil so nič! Npr.:

$$\langle S S_z k | [H, S_z] | S' S_z' k'\rangle =$$

$$= \langle S S_z k | H S_z | S' S_z' k'\rangle - \langle S S_z k | S_z H | S' S_z' k'\rangle =$$

$$= (S_z' - S_z)\langle S S_z k | H | S S_z' k'\rangle = \phi$$

$$\text{za } S_z' \neq S_z \Rightarrow \langle S S_z k | H | S' S_z' k'\rangle = \phi$$

Energiji stanj $|S S_z k)$ in $S^-|S S_z k)$ sta enaki! (enako velja za S^2 in k, $S^-|S S_z k)$ ima isti S in [k] kot $|S S_z k)$. To sle[di] iz $[T, S^-] = 0$ in $[S^2, \ldots$

$$[H, S^-] = [H, S_x - iS_y] = \phi$$

$$[H, S^-]|S S_z k) = \phi = HS^-|S S_z k) - S^- H|S S_z k) =$$

$$= HS^-|S S_z k) - S^- E(S, S_z, k)|S S_z k) =$$

$$= HS^-|S S_z k) - E(S S_z k) S^-|S S_z k) \Rightarrow HS^-|S S_z k) = E(S S_z k) S^-|\ldots$$

$$= E(S, S_z - 1, k) S^-|\ldots$$

Heisenbergov model:

produktna baza: (ta stanja že imajo dober celoten S_z!)

$S_z=2$: $|\uparrow\uparrow\uparrow\uparrow\rangle$

$S_z=1$: $|\downarrow\uparrow\uparrow\uparrow\rangle$, $|\uparrow\downarrow\uparrow\uparrow\rangle$, $|\uparrow\uparrow\downarrow\uparrow\rangle$, $|\uparrow\uparrow\uparrow\downarrow\rangle$

$S_z=0$: $|\downarrow\downarrow\uparrow\uparrow\rangle$, $|\uparrow\downarrow\downarrow\uparrow\rangle$, $|\uparrow\uparrow\downarrow\downarrow\rangle$, $|\downarrow\uparrow\uparrow\downarrow\rangle$, $|\uparrow\downarrow\uparrow\downarrow\rangle$, $|\downarrow\uparrow\downarrow\uparrow\rangle$

$S_z=-1$: $|\downarrow\downarrow\downarrow\uparrow\rangle$, $|\downarrow\downarrow\uparrow\downarrow\rangle$, $|\downarrow\uparrow\downarrow\downarrow\rangle$, $|\uparrow\downarrow\downarrow\downarrow\rangle$

$S_z=-2$: $|\downarrow\downarrow\downarrow\downarrow\rangle$

$$T|\uparrow\uparrow\uparrow\uparrow\rangle = |\uparrow\uparrow\uparrow\uparrow\rangle = e^{ika}|\uparrow\uparrow\uparrow\uparrow\rangle \Rightarrow k=$$

stanje $|\uparrow\uparrow\uparrow\uparrow\rangle$ ima tudi dober $S=2$ in $k=0$. Ostala stanja z $S=2$ dobim tako, da na stanje $|\uparrow\uparrow\uparrow\uparrow\rangle$ delujem z operatorjem S_-:

$$\left(\begin{aligned} S_-|\uparrow\rangle &= |\downarrow\rangle \\ S_-|\downarrow\rangle &= \emptyset \end{aligned}\right)$$

$$\boxed{|\uparrow\uparrow\uparrow\uparrow\rangle = |\overset{S}{2}\,\overset{S_z}{2}\,\overset{k}{0}\rangle}$$

$$S_-|\uparrow\uparrow\uparrow\uparrow\rangle = \sum_{i=1}^{4} S_{i-}|\uparrow\uparrow\uparrow\uparrow\rangle = |\downarrow\uparrow\uparrow\uparrow\rangle + |\uparrow\downarrow\uparrow\uparrow\rangle + |\uparrow\uparrow\downarrow\uparrow\rangle + |\uparrow\uparrow\uparrow\downarrow\rangle =$$

$$= S_-|220\rangle = \sqrt{2(2+1)-2(2-1)}\,|210\rangle = 2|210\rangle$$

$$\boxed{|210\rangle = \frac{1}{2}\left(|\downarrow\uparrow\uparrow\uparrow\rangle + |\uparrow\downarrow\uparrow\uparrow\rangle + |\uparrow\uparrow\downarrow\uparrow\rangle + |\uparrow\uparrow\uparrow\downarrow\rangle\right)}$$

$$S_-|210\rangle = \sqrt{2(2+1)-1(1-1)} = \sqrt{6}\,|200\rangle = \frac{1}{2}\left(\begin{aligned} &|\downarrow\downarrow\uparrow\uparrow\rangle + |\downarrow\downarrow\uparrow\uparrow\rangle + |\downarrow\uparrow\downarrow\uparrow\rangle + |\downarrow\uparrow\uparrow\downarrow\rangle \\ &+|\downarrow\uparrow\downarrow\uparrow\rangle + |\uparrow\downarrow\downarrow\uparrow\rangle + |\uparrow\downarrow\downarrow\uparrow\rangle + |\uparrow\downarrow\uparrow\downarrow\rangle \\ &+|\downarrow\uparrow\uparrow\downarrow\rangle + |\uparrow\downarrow\uparrow\downarrow\rangle + |\uparrow\uparrow\downarrow\downarrow\rangle + |\uparrow\uparrow\downarrow\downarrow\rangle \end{aligned}\right)$$

$$\boxed{|200\rangle = \frac{1}{\sqrt{6}}\left(|\downarrow\downarrow\uparrow\uparrow\rangle + |\downarrow\uparrow\downarrow\uparrow\rangle + |\downarrow\uparrow\uparrow\downarrow\rangle + |\uparrow\uparrow\downarrow\downarrow\rangle + |\uparrow\downarrow\uparrow\downarrow\rangle + |\uparrow\downarrow\downarrow\uparrow\rangle\right)}$$

itd. za stanji $|2\,{-1}\,0\rangle$ in $|2\,{-2}\,0\rangle$. Vsa ta stanja imajo enako energijo:

$$H|\uparrow\uparrow\uparrow\uparrow\rangle = J\sum_{i=1}^{4}\vec{S}_i\cdot\vec{S}_{i+1}|\uparrow\uparrow\uparrow\uparrow\rangle = J\sum_{i=1}^{4}\left[S_i^z S_{i+1}^z + \frac{1}{2}\left(S_{i-}S_{i+1,+} + S_{i+}S_{i+1,-}\right)\right]|\uparrow\uparrow\uparrow\uparrow\rangle =$$

$$= J\sum_{i=1}^{4} S_i^z S_{i+1}^z|\uparrow\uparrow\uparrow\uparrow\rangle = J\sum_{i=1}^{4}\frac{1}{2}\cdot\frac{1}{2}|\uparrow\uparrow\uparrow\uparrow\rangle = J\cdot\frac{1}{4}\cdot 4\,|\uparrow\uparrow\uparrow\uparrow\rangle = \underline{\underline{J}}\,|\uparrow\uparrow\uparrow\uparrow\rangle$$

$(\hbar=1)$

Heisenbergov model:

Iz produktnih stanj s $S_z = 1$ tvorim stanja z dobrim k:

$$T|\psi\rangle = e^{ika}|\psi\rangle$$

$$|\psi\rangle = \frac{1}{2}\left(|\downarrow\uparrow\uparrow\uparrow\rangle + e^{ika}|\uparrow\downarrow\uparrow\uparrow\rangle + e^{2ika}|\uparrow\uparrow\downarrow\uparrow\rangle + e^{3ika}|\uparrow\uparrow\uparrow\downarrow\rangle\right)$$

$$T|\psi\rangle = \frac{1}{2}\left(|\uparrow\uparrow\uparrow\downarrow\rangle + e^{ika}|\downarrow\uparrow\uparrow\uparrow\rangle + e^{2ika}|\uparrow\downarrow\uparrow\uparrow\rangle + e^{3ika}|\uparrow\uparrow\downarrow\uparrow\rangle\right) = e^{ika}|\psi\rangle$$

torej:

$$\boxed{\begin{aligned}
|1\,1\,-\tfrac{\pi}{2}\rangle &= \tfrac{1}{2}\left(|\downarrow\uparrow\uparrow\uparrow\rangle - i|\uparrow\downarrow\uparrow\uparrow\rangle - |\uparrow\uparrow\downarrow\uparrow\rangle + i|\uparrow\uparrow\uparrow\downarrow\rangle\right)\\
|1\,1\,\tfrac{\pi}{2}\rangle &= \tfrac{1}{2}\left(|\downarrow\uparrow\uparrow\uparrow\rangle + i|\uparrow\downarrow\uparrow\uparrow\rangle - |\uparrow\uparrow\downarrow\uparrow\rangle - i|\uparrow\uparrow\uparrow\downarrow\rangle\right)\\
|1\,1\,\pi\rangle &= \tfrac{1}{2}\left(|\downarrow\uparrow\uparrow\uparrow\rangle - |\uparrow\downarrow\uparrow\uparrow\rangle + |\uparrow\uparrow\downarrow\uparrow\rangle - |\uparrow\uparrow\uparrow\downarrow\rangle\right)
\end{aligned}}$$

Stanja s $S_z = 0$ in $S_z = -1$ dobim tako, da delujem na zgornja stanja z operatorjem S_-:

$$S_-|1\,1\,k\rangle = \sqrt{1(1+1) - 1(1-1)}\,|1\,1\,k\rangle = \sqrt{2}\,|1\,0\,k\rangle =$$

Spet imajo stanja z $S_z = -1, 0, 1$ enako energijo. To so enomagnonska vzbujena stanja feromagneta, torej lahko uporabimo energije magnonov, ki ste jih izpeljali na predavanjih:

$$E(S=1, S_z, k) = E(S=2, S_z, 0) - J\sin^2\frac{k}{2}$$

Zapišimo še stanja z $S_z = 0$:

$$\sqrt{2}\,|1\,0\,k\rangle = \frac{1}{2}\begin{pmatrix}
|\downarrow\downarrow\uparrow\uparrow\rangle & + e^{ika}|\downarrow\downarrow\uparrow\uparrow\rangle & + e^{2ika}|\downarrow\uparrow\downarrow\uparrow\rangle & + e^{3ika}|\downarrow\uparrow\uparrow\downarrow\rangle\\
+|\downarrow\uparrow\downarrow\uparrow\rangle & + e^{ika}|\uparrow\downarrow\downarrow\uparrow\rangle & + e^{2ika}|\uparrow\downarrow\downarrow\uparrow\rangle & + e^{3ika}|\uparrow\downarrow\uparrow\downarrow\rangle\\
+|\downarrow\uparrow\uparrow\downarrow\rangle & + e^{ika}|\uparrow\downarrow\uparrow\downarrow\rangle & + e^{2ika}|\uparrow\uparrow\downarrow\downarrow\rangle & + e^{3ika}|\uparrow\uparrow\downarrow\downarrow\rangle
\end{pmatrix} =$$

$$= \frac{1}{2}\Big((1+e^{ika})|\downarrow\downarrow\uparrow\uparrow\rangle + e^{ika}(1+e^{ika})|\uparrow\downarrow\downarrow\uparrow\rangle + e^{2ika}(1+e^{ika})|\uparrow\uparrow\downarrow\downarrow\rangle +$$
$$+ e^{3ika}(1+e^{ika})|\downarrow\uparrow\uparrow\downarrow\rangle + (1+e^{2ika})|\downarrow\uparrow\downarrow\uparrow\rangle +$$
$$+ e^{ika}(1+e^{2ika})|\uparrow\downarrow\uparrow\downarrow\rangle\Big)$$

$$\cancel{|1\,0\,k\rangle = \left[|\downarrow\downarrow\uparrow\uparrow\rangle + |\uparrow\downarrow\downarrow\uparrow\rangle e^{ik} + |\uparrow\uparrow\downarrow\downarrow\rangle e^{2ik} + |\downarrow\uparrow\uparrow\downarrow\rangle e^{3ik} + e^{ik}|\downarrow\uparrow\downarrow\uparrow\rangle + e^{3ik}|\uparrow\downarrow\uparrow\downarrow\rangle\right]}$$

$$|1\,0\,-\tfrac{\pi}{2}\rangle = \frac{1}{2}\left[|\downarrow\downarrow\uparrow\uparrow\rangle - i|\uparrow\downarrow\downarrow\uparrow\rangle - |\uparrow\uparrow\downarrow\downarrow\rangle + i|\downarrow\uparrow\uparrow\downarrow\rangle\right]$$

$$|1\,0\,\tfrac{\pi}{2}\rangle = \frac{1}{2}\left[|\downarrow\downarrow\uparrow\uparrow\rangle + i|\uparrow\downarrow\downarrow\uparrow\rangle - |\uparrow\uparrow\downarrow\downarrow\rangle - i|\downarrow\uparrow\uparrow\downarrow\rangle\right]$$

$$|1\,0\,\pi\rangle = \frac{1}{\sqrt{2}}\left[|\downarrow\uparrow\downarrow\uparrow\rangle - |\uparrow\downarrow\uparrow\downarrow\rangle\right]$$

Heisenbergov model:

Ostaneta še dve stanji s $S=0$:

Iz šestih produktnih stanj s $S_z=0$ lahko tvorim šest stanj z dobrim

$|a\rangle = \frac{1}{2}\left(|{\downarrow\downarrow\uparrow\uparrow}\rangle + |{\uparrow\downarrow\downarrow\uparrow}\rangle + |{\uparrow\uparrow\downarrow\downarrow}\rangle + |{\downarrow\uparrow\uparrow\downarrow}\rangle\right)$ $\quad q=0$

$|b\rangle = \frac{1}{2}\left(|{\downarrow\downarrow\uparrow\uparrow}\rangle + i|{\uparrow\downarrow\downarrow\uparrow}\rangle - |{\uparrow\uparrow\downarrow\downarrow}\rangle - i|{\downarrow\uparrow\uparrow\downarrow}\rangle\right)$ $\quad q=\frac{\pi}{2} \rightarrow |1\,0\,\tfrac{\pi}{2}\rangle$

$|c\rangle = \frac{1}{2}\left(|{\downarrow\downarrow\uparrow\uparrow}\rangle - |{\uparrow\downarrow\downarrow\uparrow}\rangle + |{\uparrow\uparrow\downarrow\downarrow}\rangle - |{\downarrow\uparrow\uparrow\downarrow}\rangle\right)$ $\quad q=\pi$

$|d\rangle \; \frac{1}{2}\left(|{\downarrow\downarrow\uparrow\uparrow}\rangle - i|{\uparrow\downarrow\downarrow\uparrow}\rangle - |{\uparrow\uparrow\downarrow\downarrow}\rangle + i|{\downarrow\uparrow\uparrow\downarrow}\rangle\right)$ $\quad q=-\frac{\pi}{2} \rightarrow |1\,0\,-\tfrac{\pi}{2}\rangle$

$|e\rangle \; \frac{1}{\sqrt{2}}\left(|{\downarrow\uparrow\downarrow\uparrow}\rangle + |{\uparrow\downarrow\uparrow\downarrow}\rangle\right)$ $\quad q=0$

$|f\rangle = \frac{1}{\sqrt{2}}\left(|{\downarrow\uparrow\downarrow\uparrow}\rangle - |{\uparrow\downarrow\uparrow\downarrow}\rangle\right)$ $\quad q=\pi \rightarrow |1\,0\,\pi\rangle$

Iz stanj $|a\rangle$ in $|e\rangle$ lahko tvorim stanje $|200\rangle$:

$$|200\rangle = \frac{2|a\rangle + \sqrt{2}|e\rangle}{\sqrt{6}}$$

Nanj ortogonalno je stanje $|000\rangle$

$$\boxed{|000\rangle = \frac{\sqrt{2}|a\rangle - 2|e\rangle}{\sqrt{6}} = \frac{1}{\sqrt{12}}\left[|{\downarrow\downarrow\uparrow\uparrow}\rangle + |{\uparrow\downarrow\downarrow\uparrow}\rangle + |{\uparrow\uparrow\downarrow\downarrow}\rangle + |{\downarrow\uparrow\uparrow\downarrow}\rangle\right] - \frac{1}{\sqrt{6}}\left[|{\downarrow\uparrow\downarrow\uparrow}\rangle + \right.}$$

Stanje $|c\rangle = \boxed{|00\pi\rangle = \frac{1}{2}\left[|{\downarrow\downarrow\uparrow\uparrow}\rangle - |{\uparrow\downarrow\downarrow\uparrow}\rangle + |{\uparrow\uparrow\downarrow\downarrow}\rangle - |{\downarrow\uparrow\uparrow\downarrow}\rangle\right]}$

Še energiji teh dveh stanj:

$$H|{\downarrow\downarrow\uparrow\uparrow}\rangle = J\cdot\cancel{\left(\tfrac{1}{4} - \tfrac{1}{4} + \tfrac{1}{4} - \tfrac{1}{4}\right)}|{\downarrow\downarrow\uparrow\uparrow}\rangle + \tfrac{1}{2}J\left(|{\downarrow\uparrow\downarrow\uparrow}\rangle + |{\uparrow\downarrow\uparrow\downarrow}\rangle\right)$$

$$H|{\uparrow\downarrow\downarrow\uparrow}\rangle = \quad -\|- \quad + \tfrac{1}{2}J\left(|{\downarrow\uparrow\downarrow\uparrow}\rangle + |{\uparrow\downarrow\uparrow\downarrow}\rangle\right)$$

$$H|{\uparrow\uparrow\downarrow\downarrow}\rangle = \quad -\|- \quad + \tfrac{1}{2}J\left(|{\downarrow\uparrow\downarrow\uparrow}\rangle + |{\uparrow\downarrow\uparrow\downarrow}\rangle\right)$$

$$H|{\downarrow\uparrow\uparrow\downarrow}\rangle = \quad -\|- \quad + \tfrac{1}{2}J\left(|{\uparrow\downarrow\uparrow\downarrow}\rangle + |{\downarrow\uparrow\downarrow\uparrow}\rangle\right)$$

$$H|{\uparrow\downarrow\uparrow\downarrow}\rangle = J\left(-\tfrac{1}{4} - \tfrac{1}{4} - \tfrac{1}{4} - \tfrac{1}{4}\right)|{\uparrow\downarrow\uparrow\downarrow}\rangle + \tfrac{1}{2}J\left(|{\downarrow\uparrow\uparrow\downarrow}\rangle + |{\uparrow\uparrow\downarrow\downarrow}\rangle + |{\uparrow\downarrow\downarrow\uparrow}\rangle + |{\downarrow\downarrow\uparrow\uparrow}\rangle\right)$$

$$H|{\downarrow\uparrow\downarrow\uparrow}\rangle = J\left(-\tfrac{1}{4} + \tfrac{1}{4} - \tfrac{1}{4} - \tfrac{1}{4}\right)|{\downarrow\uparrow\downarrow\uparrow}\rangle + \tfrac{1}{2}J\left(\quad -\|- \quad\right)$$

torej: $H|a\rangle = \sqrt{2}J|e\rangle$

$H|e\rangle = -J|e\rangle + \sqrt{2}J|a\rangle$

$$\boxed{\begin{aligned} H\sqrt{2}|e\rangle &= -J\sqrt{2}|e\rangle + \tfrac{1}{2}J\cdot 4|a\rangle \\ H|e\rangle &= -J|e\rangle + \tfrac{2J}{\sqrt{2}}|a\rangle \\ H\,2|a\rangle &= \tfrac{1}{2}J\cdot 4\cdot\sqrt{2}|e\rangle \rightarrow H|a\rangle = \sqrt{2}J|e\rangle \end{aligned}}$$

Heisenbergov model:

$$\begin{pmatrix} \langle a|H|a\rangle & \langle a|H|e\rangle \\ \langle e|H|a\rangle & \langle e|H|e\rangle \end{pmatrix} = \begin{pmatrix} 0, & \sqrt{2}J \\ \sqrt{2}J, & -J \end{pmatrix}$$

lastne vrednosti: $-E(-J-E) - 2J^2 = 0$

$$JE + E^2 - 2J^2 = 0$$

$$E^2 + JE - 2J^2 = 0$$

$$\begin{pmatrix} -J, & \sqrt{2}J \\ \sqrt{2}J, & -2J \end{pmatrix}\begin{pmatrix} 2 \\ \sqrt{2} \end{pmatrix} = 0$$

$$E = \frac{-J \pm \sqrt{J^2 + 8J^2}}{2} = \frac{-J \pm 3J}{2} = J, -2J$$

$$\begin{pmatrix} 2J & \sqrt{2}J \\ \sqrt{2}J & J \end{pmatrix}\begin{pmatrix} \sqrt{2} \\ -2 \end{pmatrix}$$

$E = J \qquad |\psi\rangle = \dfrac{2|a\rangle + \sqrt{2}|e\rangle}{\sqrt{6}} = |200\rangle$

$E = -2J \qquad |\psi\rangle = \dfrac{\sqrt{2}|a\rangle - 2|e\rangle}{\sqrt{6}} = |000\rangle$

$J > 0$ (antiferomagnet)

osnovno stanje: $E = -2J$

$$|\psi\rangle = \frac{1}{\sqrt{12}}\left[|\downarrow\downarrow\uparrow\uparrow\rangle + |\uparrow\downarrow\downarrow\uparrow\rangle + |\uparrow\uparrow\downarrow\downarrow\rangle + |\downarrow\uparrow\uparrow\downarrow\rangle\right] - \frac{1}{\sqrt{6}}\left(|\uparrow\downarrow\uparrow\downarrow\rangle + |\downarrow\uparrow\downarrow\uparrow\rangle\right)$$

$J < 0$ (Feromagnet)

osnovno stanje: $E = J$

$|\psi\rangle = |\uparrow\uparrow\uparrow\uparrow\rangle = |220\rangle$ 5x degenerirano

stanja $|2 S_z 0\rangle$ imajo vsa enako energ