Dinamika

Newtonovi zakoni (Isaac Newton 1642–1727)

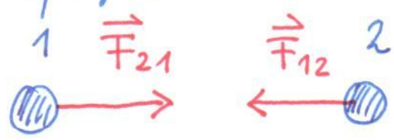
1.) $\sum_i \vec{F}_i = 0 \Rightarrow \vec{v}$: konstantna

Telo miruje oziroma se giblje premo enakomerno, če je vsota vseh sil, ki delujejo na telo, enaka 0.

$$\sum_i \vec{F}_i \equiv \vec{F}_1 + \vec{F}_2 + \vec{F}_3 + \dots$$

2.) $\sum_i \vec{F}_i = m\,\vec{a}$

Vsota vseh sil, ki delujejo na telo, je enaka masi krat pospešek telesa.

3.) 

$\vec{F}_{12} + \vec{F}_{21} = 0$

Če deluje prvo telo na drugo z neko silo, deluje drugo telo na prvo z nasprotno enako silo.

— o —

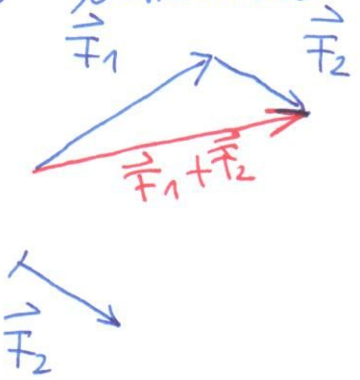
Masa: m [kg] je skalar (ni vektor) lahko bi tenzor

- Velja zakon o ohranitvi mase (tudi makroskopskih teles, kemijske reakcije)

Ne velja v primeru jedrskih reakcij, tudi mikroskopskih delcev

Sila: $\vec{F}$; $F\ [N = \frac{kg\,m}{s^2}]$ je vektor

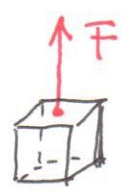
- Sile se vektorsko seštevajo:



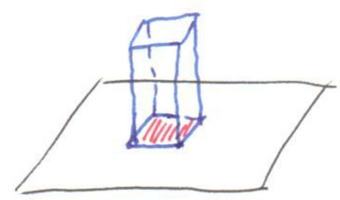
Vektorje lahko vzporedno premikamo, rezultat vektorskih operacij ostaja nespremenjen

- Sile lahko delujejo

a.) točkovno



b.) površinsko (ploskovno)



c.) volumsko

- Sila teže $\vec{F}_g = m\vec{g}$ deluje na vsa telesa, ki se nahajajo v gravitacijskem polju (primer na površini Zemlje).

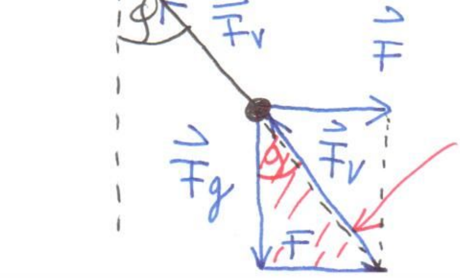
1) Newtonov zakon $\sum_i \vec{F}_i = 0 \Rightarrow \vec{v} = \text{konst}$

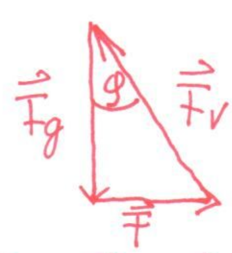
- Zgled:

$$\frac{F}{F_g} = tg\,\varphi$$

$$\boxed{F = F_g\,tg\,\varphi}$$

$$F_g = F_v \cdot \cos\varphi$$

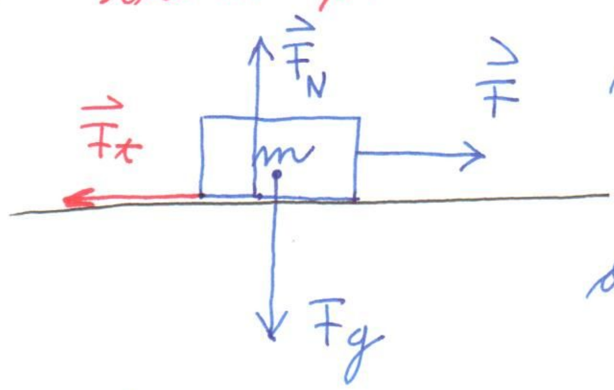




$\vec{F}_g + \vec{F}_V + \vec{F} = 0$!

Tvorijo zaključen trikotnik

2. Zgled Sila trenja:



v = konstantna

$$\vec F + \vec F_t = 0$$

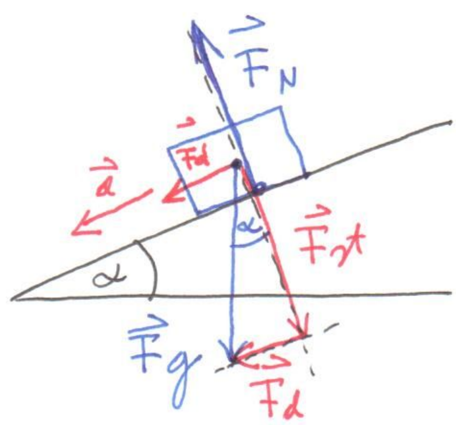
ali $F_t = F$ po velikosti!

$F_t = F_g \cdot k_t$ (velja za ravno podlago!)

k_t: koeficient trenja. Nima enote [1].

2.) $\sum_i \vec F_i = m\vec a$

- Zgled: sile na klancu



Silo teže razstavimo na dve komponenti: $\vec F_d$ in $\vec F_{st}$

$$\vec F_g = \vec F_d + \vec F_{st}$$

$$\vec F_N + \vec F_{st} = 0$$

$F_d = F_g \sin\alpha = mg\sin\alpha$

$F_{st} = F_g \cos\alpha = mg\cos\alpha$

Ker telo drsi vzdolž klanca in se ne pogreza v klanec!

$$\sum_i \vec F_i = \vec F_d + \underbrace{\vec F_{st} + \vec F_N}_{=0} = \boxed{\vec F_d = m\vec a}$$

Telo se giblje enakomerno pospešeno, $\vec a$ je vektor, ki kaže vzporedno s klancem!

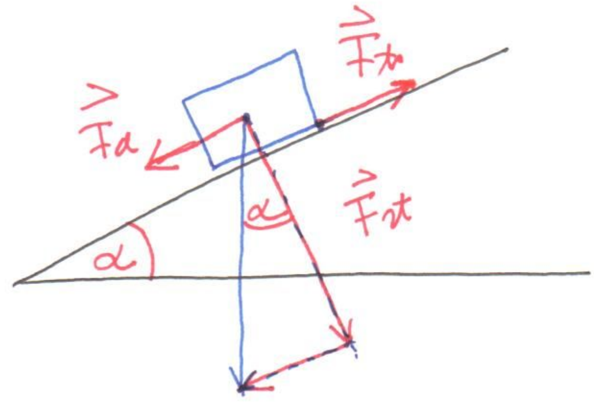
$$mg\sin\alpha = ma$$

$$a = g\sin\alpha$$

Zgled 2: sile na klancu s trenjem:

$F_t = F_{nt} \cdot k_t$!

$F_t = mg\cos\alpha \cdot k_t$



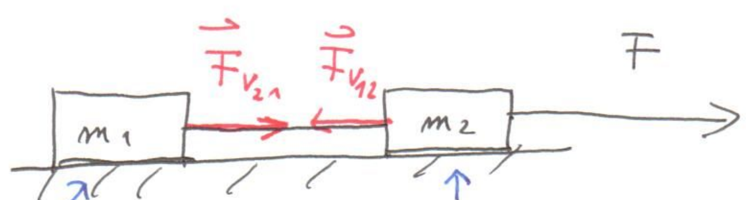
$$\vec{F}_d + \vec{F}_t = m\vec{a}$$

$$F_d - F_t = ma$$

$$mg\sin\alpha - mg\cos\alpha\, k_t = ma$$

$$a = g(\sin\alpha - k_t\cos\alpha)$$

3.) o⟶ ⟵o Zgled za uporabo 3. Newt. zakona:



$$|\vec{F}_{12}| = |\vec{F}_{21}| = F_v$$

$$\vec{F}_{12} + \vec{F}_{21} = 0$$

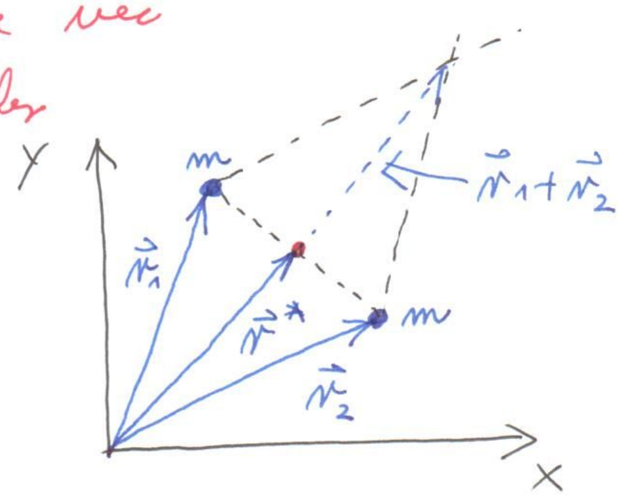
$F_v = m_1 \cdot a$ $\qquad$ $F - F_v = m_2 a$

$$F - m_1 a = m_2 a \Rightarrow a = \frac{F}{m_1 + m_2}$$

$$F_v = F\,\frac{m_1}{m_1 + m_2}$$

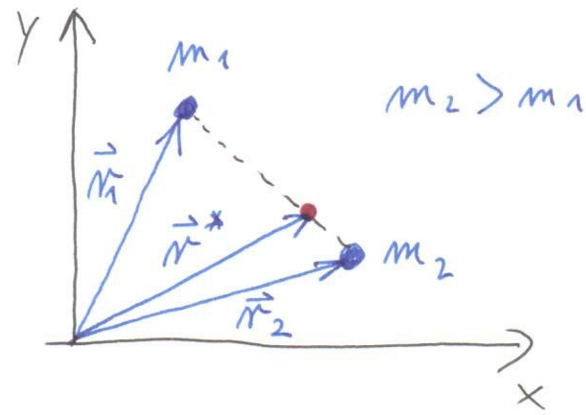
Težišče sistema več točkastih teles

a.) Težišče sistema dveh teles z enakima masama



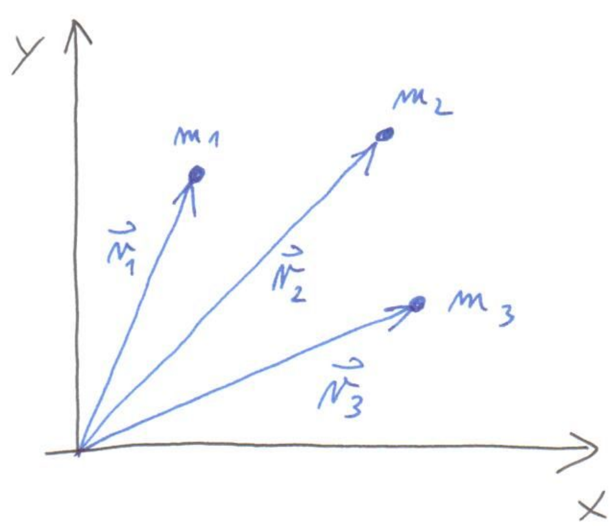
$$\vec{r}^* = \frac{1}{2}\left(\vec{r}_1 + \vec{r}_2\right)$$

b.) Težišče sistema dveh teles različnih mas:



$$\vec{r}^* = \frac{m_1\vec{r}_1 + m_2\vec{r}_2}{m_1 + m_2}$$

c.) Težišče sistema več teles



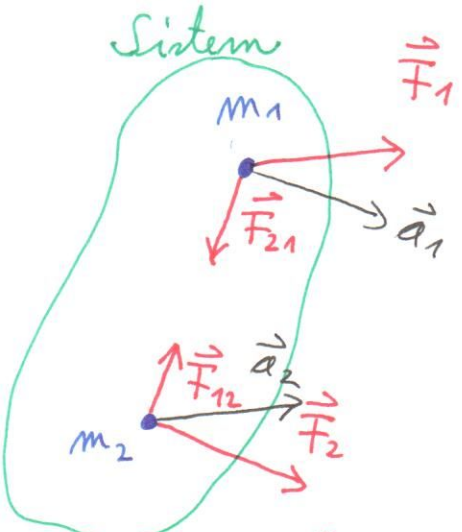
$$\vec{r}^* = \frac{\sum_i m_i \vec{r}_i}{\sum_i m_i}$$

Oziroma

$$x^* = \frac{\sum_i m_i x_i}{\sum_i m_i} \; ; \; y^* = \frac{\sum_i m_i y_i}{\sum_i m_i} \; ; \; \ldots$$

Izrek o gibanju težišča sistema točkastih teles

Za posamezno točkasto telo velja: $\sum_i \vec{F}_i = m\vec{a}$



$\vec{F}_1$: sila (vsota sil), ki deluje na prvo telo od zunaj (od teles iz okolice)

$\vec{F}_2$: kot ↑, le na drugo telo

$\vec{F}_{12}$: sila prvega telesa na drugo telo

F_{21}: obratno kot ↑.

Pozor: $\vec{F}_{12} + \vec{F}_{21} = 0$

3. Newt. zakon!

$$\vec{F}_1 + \vec{F}_{21} = m_1\vec{a}_1$$

$$\vec{F}_2 + \vec{F}_{12} = m_2\vec{a}_2$$

$\vec{a}_1$ in $\vec{a}_2$ sta pospeška prvega in drugega telesa

$$\underbrace{F_1 + F_2}_{\text{vsota vseh zunanjih sil}} + 0 = m_1\vec{a}_1 + m_2\vec{a}_2 = \underbrace{\frac{m_1\vec{a}_1 + m_2\vec{a}_2}{m_1+m_2}}_{\vec{a}^*} \underbrace{(m_1+m_2)}_{\text{vsota vseh mas}}$$

$\vec{a}^*$ pospešek težišča sistema

vsota po vseh zunanjih silah

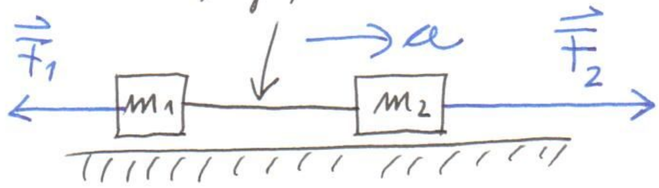
$$\sum_{i=1}^{N_i} \vec{F}_i = \frac{\sum_j m_j a_j}{\sum_j m_j} \sum_j m_j = \vec{a}^* M$$

pospešek težišča

masa celotnega sistema!

Primeri:

a.)



$F_2 > F_1$

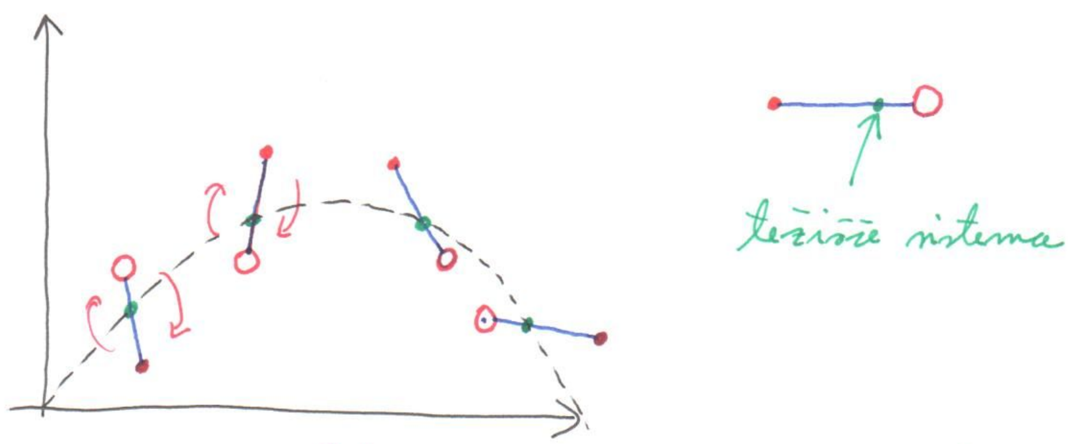
$$\vec{F}_1 + \vec{F}_2 = (m_1 + m_2) \cdot a$$

Obe telesi se gibljeta z istim pospeškom!

$$F_2 - F_1 = (m_1 + m_2)\, a$$

$$a = \frac{F_2 - F_1}{(m_1 + m_2)}$$

b.)



Togo telo se lahko med poševnim metom vrti, a težišče se giblje po paraboli!

Gravitacijska sila

$$F = \varkappa \frac{m M}{r^2}$$

$\varkappa$: Gravitacijska konstanta: $6{,}67 \cdot 10^{-11} \frac{Nm^2}{kg^2}$

Primer: Izračunaj orbito geostacionarnega satelita.

$r_Z = 6{,}4 \cdot 10^6\, m$ (Radij Zemlje)

Geostacionarni satelit kroži v ekvatorialni ravnini z isto ω kot Zemlja

težnostni pospešek na površini Zemlje

$$F = \varkappa \frac{m M}{r^2} = \varkappa \frac{M m}{r_Z^2} \frac{r_Z^2}{r^2} = g_0 \cdot m \frac{r_Z^2}{r^2}$$

$$F(r) = m\, g(r) = m\, g_0 \frac{r_Z^2}{r^2}$$

$g(r) = \omega^2 \cdot r$ = radialni pospešek,

ω: kotna hitrost Zemlje

$$g_0 \frac{r_Z^2}{r^2} = \omega^2 \cdot r \Rightarrow r^3 = g_0 \frac{r_Z^2}{\omega^2}$$

$$r^3 = 9{,}8 \frac{m}{s^2} \frac{(6{,}4 \cdot 10^3)^2 m^2 \cdot (3600 \cdot 24)^2 s^2}{4\pi^2} \sim \underline{\underline{42.000\, km}}$$

Izrek o gibalni količini

$$\vec{F} = m\vec{a}\ ; \quad \vec{F} = m\frac{d\vec{v}}{dt} \Rightarrow \vec{F}dt = m\,d\vec{v}$$

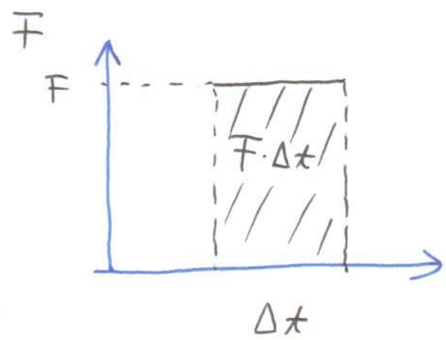
sunek sile   Gibalna količina

$$\int_0^t \vec{F}dt = \int_{\vec{v}_z}^{\vec{v}_k} m\,d\vec{v} = m\vec{v}_k - m\vec{v}_z$$

$$= \vec{G}_k - \vec{G}_z$$

$$\left(\int F_x dt, \int F_y dt, \int F_z dt\right) = \left(\int_{v_{xz}}^{v_{xk}} m\,dv_x, \int_{v_{yz}}^{v_{yk}} m\,dv_y, \dots\right)$$

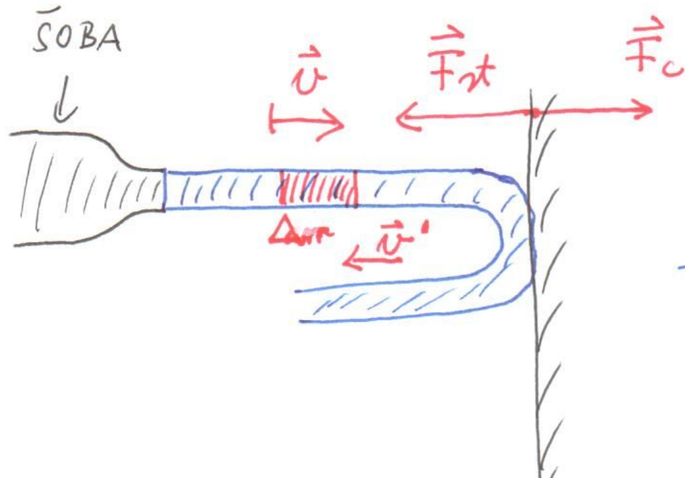
$$\boxed{\text{Sunek sile: } \int F dt \ [\mathrm{Ns}]}$$



$$\int_0^t \vec{F}dt = \vec{G}_k - \vec{G}_z = m\vec{v}_k - m\vec{v}_z$$

Sunek sile je enak spremembi gibalne količine

Sila curka



sunek sile

sprememba $\vec{G}$ curka

čas trajanja interakcije s steno

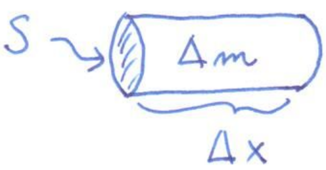
$$\vec{F}_{st}\,\Delta t = \Delta m\,\vec{v}\,' - \Delta m\,\vec{v}$$

$$-F_{st} = \frac{\Delta m}{\Delta t}\left(-v' - v\right)$$

$$F_{st} = \phi_m\,(v + v')$$

$$F_{st} = \boxed{F_c = S\rho v\,(v + v')}$$

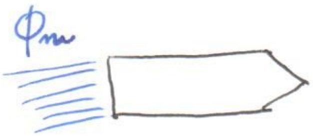
Sila curka



$$\Delta m = S\,\Delta x\cdot\rho \quad ; \quad \frac{\Delta m}{\Delta t} = \phi_m = S\rho\,\frac{\Delta x}{\Delta t} = S\rho v$$

hitrost v curka pred udarom na steno

Primer: raketa



$$F = \phi_m \cdot v_c = m\,a$$

masni pretok

hitrost curka glede na raketo

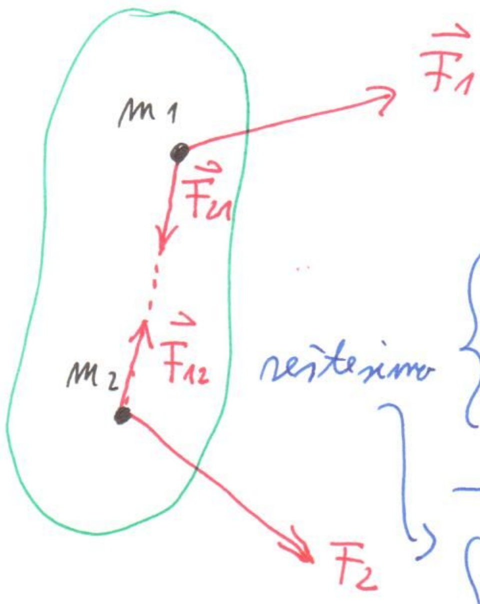
Problem: $m = m(t) = m_0 - \phi_m\cdot t$

Upoštevati moramo, da se masa rakete manjša zaradi iztekanja goriva (izgorevanja).

$$\phi_m v_c = m(t)\,\frac{dv}{dt} \Rightarrow v = v_c\,\phi_m \int_0^t \frac{dt}{m_0 - \phi_m t}$$

$$v = v_c\left(-\ln(m_0 - \phi_m t)\right)\Big|_0^t = v_c \ln\frac{m_0}{m_0 - \phi_m\cdot t}$$

Izrek o gibalni količini sistema več točkastih teles



končna hitrost prvega

začetna hitrost

$$\int \vec{F}_1\,dt + \int \vec{F}_{21}\,dt = m_1 \vec{v}_{k1} - m_1 \vec{v}_{z1}$$

seštejemo

$$\int \vec{F}_2\,dt + \int \vec{F}_{12}\,dt = m_2 \vec{v}_{k2} - m_2 \vec{v}_{z2}$$

$$\int \vec{F}_1\,dt + \int \vec{F}_2\,dt = m_1 \vec{v}_{k1} + m_2 \vec{v}_{k2} - (m_1 \vec{v}_{z1} + m_2 \vec{v}_{z2})$$

III. N. Z.:

$\vec{F}_{21} + \vec{F}_{12} = 0$ za $\forall t$

$\Rightarrow \int \vec{F}_{21}\,dt + \int \vec{F}_{12}\,dt = 0$

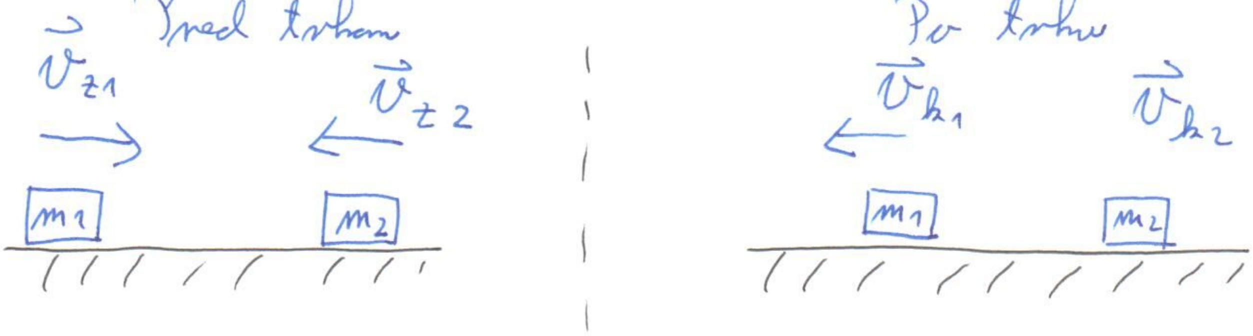
$$\sum_i \int \vec{F}_i\,dt = \sum_j (m_j \vec{v}_{kj} - m_j \vec{v}_{zj})$$

vsota sunkov vseh zunanjih sil

sprememba celotne gibalne količine sistema

$$\vec{G} = \sum_j m_j \vec{v}_j$$: gibalna količina sistema toč. teles je vsota posameznih gibalnih količin!

Trki dveh teles



Med trkom ter v času opazovanja velja:

$$\sum_i \int F_i \, dt = 0$$

Na sistem ne delujejo zunanje sile oziroma: vsota sunkov vseh zunanjih sil $= 0 \Rightarrow$

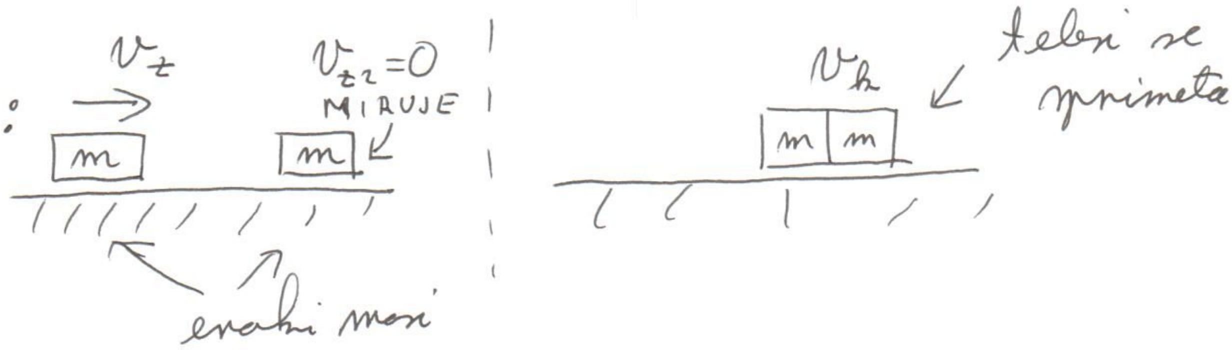
$$\sum_j m_j \vec v_{zj} = \sum_j m_j \vec v_{kj}$$

Ohranitev gibalne količine sistema!

Za dve telesi velja:

$$m_1 \vec v_{z1} + m_2 \vec v_{z2} = m_1 \vec v_{k1} + m_2 \vec v_{k2}$$

Primer:



$$m v_z + 0 = m v_k + m v_k \Rightarrow v_k = \frac{m v_z}{2m} = \frac{v_z}{2}$$

Izrek o kinetični energiji in delu

$$F = ma = m\frac{dv}{dt}\quad ; \quad \text{Množim z } dx : \text{diferencial poti}$$

$$F\,dx = m\frac{dv\,dx}{dt} = m\,v\,dv$$

" v "

$$\int_{x_z}^{x} F\,dx = \int_{v_z}^{v_k} m\,v\,dv = m\frac{v^2}{2}\Bigg|_{v_z}^{v_k} = m\frac{v_k^2}{2} - m\frac{v_z^2}{2}$$

$$A = \int_{x_z}^{x_k} F\,dx = \underbrace{m\frac{v_k^2}{2} - m\frac{v_z^2}{2}}_{\text{Sprememba kinetične energije}} = W_{kin}^{k} - W_{kin}^{z}$$

Delo sile

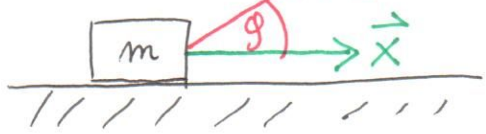
Različne oblike zapisa za delo:

a.) F : konstantna $\Rightarrow \quad A = F\cdot\Delta x$ oziroma:

$$A = \int_{x_z}^{x_k} F\,dx = F\int_{x_z}^{x_k} dx = F\,(x_k - x_z) = F\cdot\Delta x$$

Delo : sila * pot

$A\ [Nm = J]\quad J$: Joule enota za delo b.)

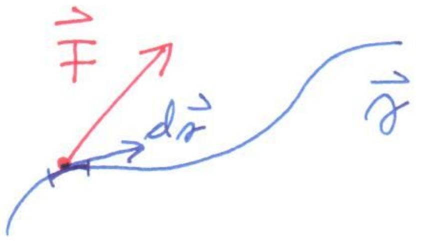


dolžina poti

$$A = \vec{F}\cdot\vec{x} = F\,x\cos\varphi \quad (27)$$

Smer sile in smer poti sta pod kotom φ, $\vec{F}$ še vedno konstanta

c.)



Pot na nekem ukrivljenem tiru

$$\boxed{A = \int\limits_{\text{po tiru gibanja}} \vec{F}\,d\vec{r}}$$

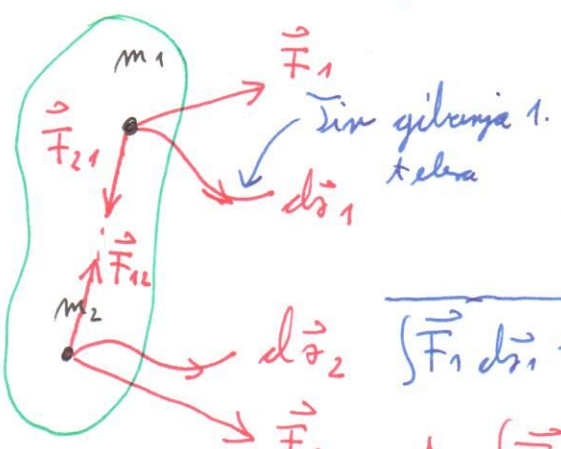
Najbolj splošna enačba

Moč

$$P = \frac{dA}{dt} \quad \left[W = \frac{J}{s} \right]$$

Moč: delo na časovno enoto.

Izrek o kinetični energiji sistema več točkastih teles



$$\int \vec{F}_1\,d\vec{r}_1 + \int \vec{F}_{21}\,d\vec{r}_1 = m_1\frac{v_{k1}^2}{2} - m_1\frac{v_{z1}^2}{2}$$

$$\int \vec{F}_2\,d\vec{r}_2 + \int \vec{F}_{12}\,d\vec{r}_2 = m_2\frac{v_{k2}^2}{2} - m_2\frac{v_{z2}^2}{2}$$

$$\int \vec{F}_1\,d\vec{r}_1 + \int \vec{F}_2\,d\vec{r}_2 + \int \vec{F}_{21}\,d\vec{r}_1 + \int \vec{F}_{12}\,d\vec{r}_2 = m_1\frac{v_{k1}^2}{2} + m_2\frac{v_{k2}^2}{2} - \left(m_1\frac{v_{z1}^2}{2} + m_2\frac{v_{z2}^2}{2}\right)$$

$A = \int \vec{F}_1\,d\vec{r}_1 + \int \vec{F}_2\,d\vec{r}_2$: delo zunanjih sil

$A_n = \int \vec{F}_{21}\,d\vec{r}_1 + \int \vec{F}_{12}\,d\vec{r}_2$: delo notranjih sil

Delo vseh sil (zunanjih in notranjih) je enako spremembi celotne kinetične energije

$$A + A_n = m_1\frac{v_{k1}^2}{2} + m_2\frac{v_{k2}^2}{2} - \left(m_1\frac{v_{z1}^2}{2} + m_2\frac{v_{z2}^2}{2}\right) = W^k_{kin1} + W^k_{kin2} - W^z_{kin1} - W^z_{kin2}$$

Prožni trk dveh teles

- Pri prožnem trku se ohranja kinetična energija sistema ker je $A=0$ ter $\boxed{A_n=0}$ Pomembno: delo notranjih sil $A_n=0$!

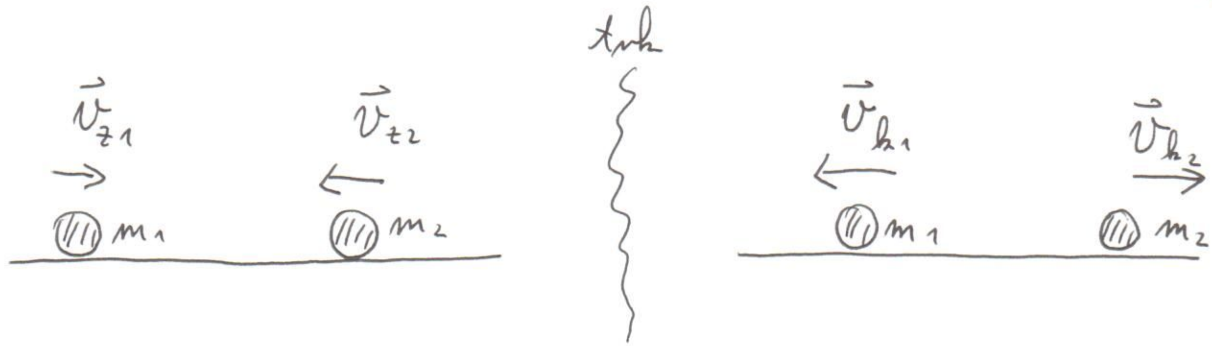
Ohranitev kinetične energije:

$$(1) \quad m_1 \frac{v_{z1}^2}{2} + m_2 \frac{v_{z2}^2}{2} = m_1 \frac{v_{k1}^2}{2} + m_2 \frac{v_{k2}^2}{2}$$

Ohranitev gibalne količine:

$$(2) \quad m_1 \vec{v}_{z1} + m_2 \vec{v}_{z2} = m_1 \vec{v}_{k1} + m_2 \vec{v}_{k2}$$

Primer:



$$m_1 v_{z1} + m_2 v_{z2} = m_1 v_{k1} + m_2 v_{k2} \quad (2)$$

$$m_1 v_{z1}^2 + m_2 v_{z2}^2 = m_1 v_{k1}^2 + m_2 v_{k2}^2 \quad (1) \quad / : \text{enačbo smo množili z } 2$$

$$\left.\begin{aligned} v_{z1} - v_{k1} &= \frac{m_2}{m_1}(v_{k2} - v_{z2}) \\ v_{z1}^2 - v_{k1}^2 &= \frac{m_2}{m_1}(v_{k2}^2 - v_{z2}^2) \end{aligned}\right\} : \qquad ; \quad \frac{m_2}{m_1} = \eta$$

$$v_{k2}^2 - v_{z2}^2 \Rightarrow (v_{k2} - v_{z2})(v_{k2} + v_{z2})$$

$$\left.\begin{aligned} v_{z1} + v_{k1} &= v_{k2} + v_{z2} \\ v_{z1} - v_{k1} &= \eta(v_{k2} - v_{z2}) \end{aligned}\right\} +$$

$$2 v_{z1} = v_{k2}(1+\eta) + v_{z2}(1-\eta)$$

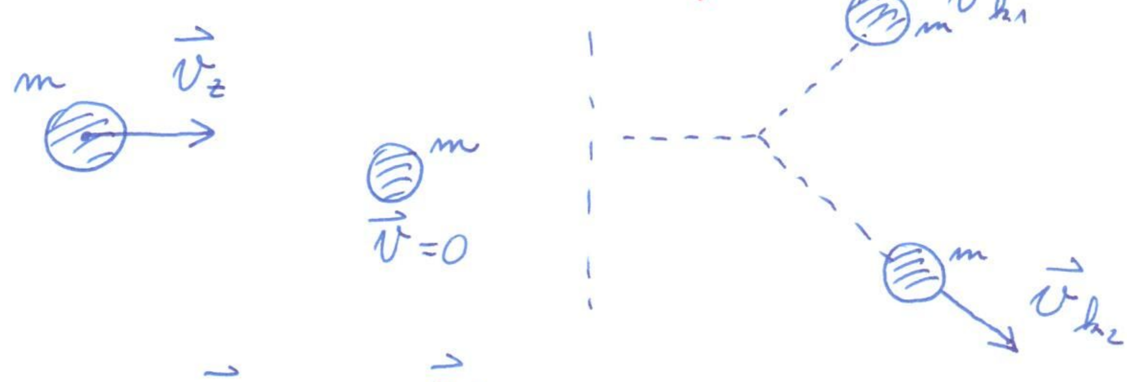
$$v_{k2} = \frac{2 v_{z1} + (\eta - 1) v_{z2}}{1+\eta} = \frac{2 v_{z1} + \left(\frac{m_2}{m_1} - 1\right) v_{z2}}{1 + \frac{m_2}{m_1}}$$

Primer: $m_1 >> m_2 \Rightarrow \eta = 0$

ter $v_{z2} = -v_{z1} \Rightarrow v_{k2} = \frac{2v_{z1} - (-v_{z1})}{1} = \underline{\underline{3 v_{z1}}}$

Pri prožnem trku dveh teles z zelo različnima masama (težko in lahko), ki gresta v začetku eden proti drugemu z nasprotno enakima hitrostima, se po odboju lahko telo giblje s 3× tno začetno hitrostjo v nasprotno smer.

Prožni trk dveh enakih kroglic: v 2D!



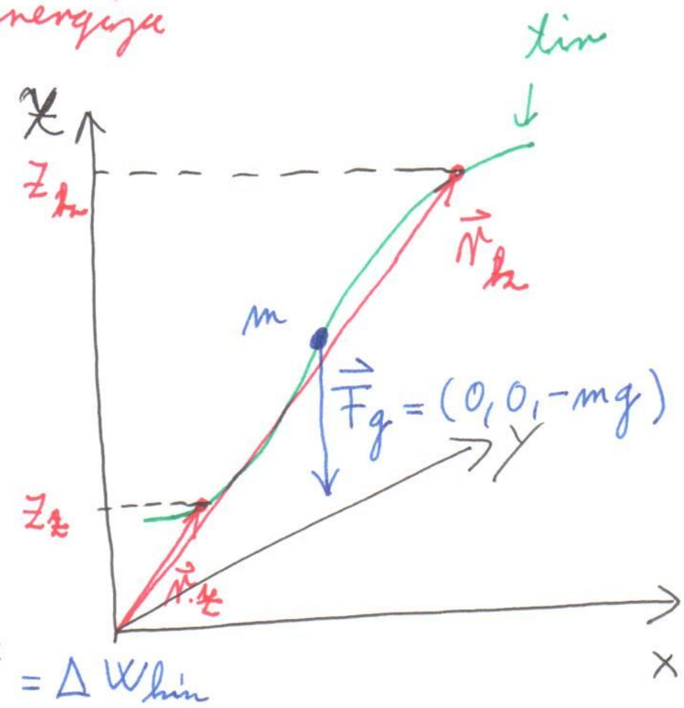
$$m\vec{v}_z = m\vec{v}_{k1} + m\vec{v}_{k2}$$

$$m\frac{v_z^2}{2} = m\frac{v_{k1}^2}{2} + m\frac{v_{k2}^2}{2}$$

Potencialna energija

Telo se giblje po tirnici:

Izrek o kinetični energiji:



$$A = W_{kin}^{k} - W_{kin}^{z}$$

$$A = A_0 + A_{Fg} = W_{kin}^{k} - W_{kin}^{z} = \Delta W_{kin}$$

delo vseh sil RAZEN! sile teže

delo sile teže

$$A_0 = \Delta W_{kin} - A_{Fg}$$

$$A_{Fg} = \int_{\vec{r}_z}^{\vec{r}_k} \vec{F}_g\, d\vec{r} = \int_{\vec{r}_z}^{\vec{r}_k} (0,0,-mg)\cdot(dx, dy, dz) =$$

po tirnici od $\vec{r}_z$ do $\vec{r}_k$

$$= -\int_{z_z}^{z_k} mg\, dz = -mg\,(z_k - z_z)$$

ali z alternativnimi oznakami

$$-A_{Fg} = mg\,(z_k - z_z) = mgh_k - mgh_z$$

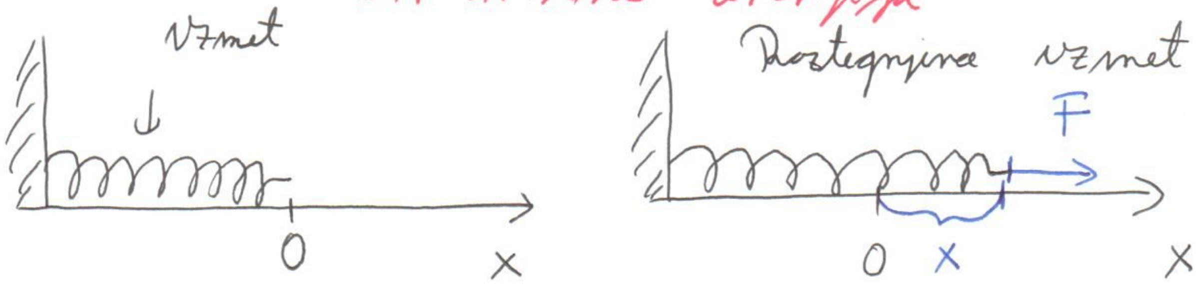
Izrek o kinetični in potencialni energiji

$$A_0 = W_{kin}^{k} - W_{kin}^{z} + W_{pot}^{k} - W_{pot}^{z}$$

$W_{pot} = mgh$: Potencialna energija

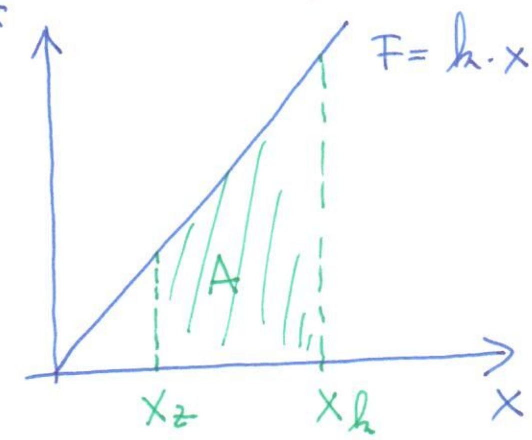
A_0 : delo vseh sil razen sile teže!

Prožnostna energija



$F = k \cdot x$; k: koeficient vzmeti $\left[\frac{N}{m}\right]$

Delo pri raztezanju vzmeti:



$$A = \int_{x_z}^{x_k} F \, dx = \int_{x_z}^{x_k} k x \, dx =$$

$$= k \frac{x^2}{2} \Big|_{x_z}^{x_k} = k \frac{x_k^2}{2} - k \frac{x_z^2}{2} = W_{pr}^{k} - W_{pr}^{z}$$

Prožnostna energija: $W_{pr} = k \frac{x^2}{2}$

Izrek o kinetični, potencialni in prožnostni energiji:

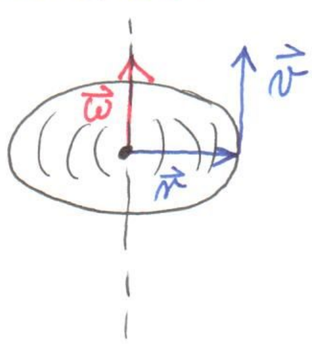
$$A_o = \Delta W_{kin} + \Delta W_{pot} + \Delta W_{pr}$$

delo vseh sil razen sile teže ter sile vzmeti (katere prispevek je upoštevan v ΔW_{pr})

Primer: Bungee jumping

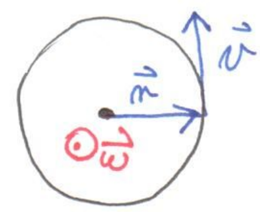
Navor, vrtenje točkastega telesa

Ponovitev:



$$\vec{v} = \vec{\omega} \times \vec{r}$$

$$\vec{a}_t = \vec{\alpha} \times \vec{r}$$

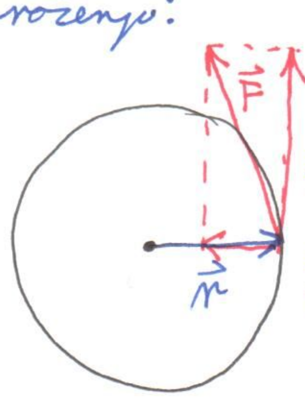


Pospešeno kroženje:

$$\sum_i \vec{F}_i = m\vec{a} \; ; \; \vec{a} = \vec{a}_r + \vec{a}_t$$

radialni pospešek

tangencialni pospešek



$$\vec{F} = \vec{F}_t + \vec{F}_r$$

$\vec{F}_t$: tangencialna sila

$\vec{F}_r$: radialna sila

$$\left(\vec{F}_t + \vec{F}_r\right) = \vec{F} = m\left(\vec{a}_r + \vec{a}_t\right) \;/\; \vec{r}\times$$

$$\left(\vec{r}\times\vec{F}_t + \cancel{\vec{r}\times\vec{F}_r}^{\,0}\right) = m\left(\cancel{\vec{r}\times\vec{a}_r}^{\,0} + \vec{r}\times\vec{a}_t\right) = \vec{r}\times\vec{F}$$

$$\vec{a}\times(\vec{b}\times\vec{c}) = \vec{b}\,(\vec{a}\cdot\vec{c}) - \vec{c}\,(\vec{a}\cdot\vec{b})$$

$$\underline{\vec{r}\times\vec{a}_t} = \vec{r}\times(\vec{\alpha}\times\vec{r}) =$$

$$= \vec{\alpha}\cdot(\vec{r}\cdot\vec{r}) - \vec{r}\,\cancel{(\vec{\alpha}\cdot\vec{r})}^{\,0} = \underline{r^2\cdot\vec{\alpha}}$$

$$\vec{M} = \vec{r}\times\vec{F} = m r^2 \vec{\alpha} = J\cdot\vec{\alpha}$$

Navor sile

vztrajnostni moment

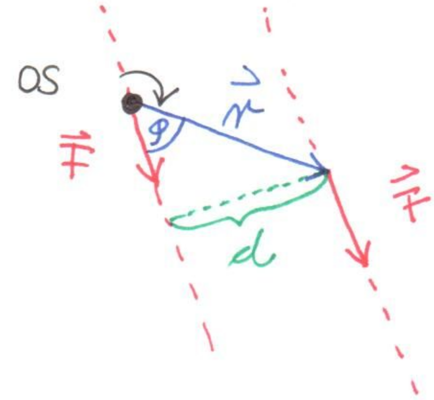
kotni pospešek

Newtonov zakon za vrtenje točkastega telesa: $\boxed{\vec{M} = J\cdot\vec{\alpha}}$

Velikost navora $|\vec{M}| = M$

enota:

$M\ [Nm]$

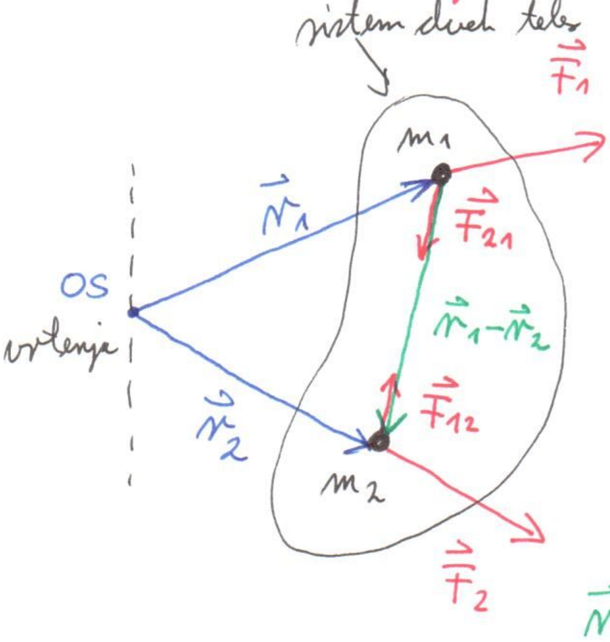


$$|\vec{M}| = |\vec{r} \times \vec{F}| =$$

$$= r \cdot F \cdot \sin\varphi = F \cdot d$$

$$M = rF\sin\varphi = F \cdot d$$

Vrtenje sistema točkastih teles (togega telesa)



$$\vec{r}_1 \times (\vec{F}_1 + \vec{F}_{21}) = m_1 (\vec{r}_1 \times \vec{a}_1)$$

$$\vec{r}_2 \times (\vec{F}_2 + \vec{F}_{12}) = m_2 (\vec{r}_2 \times \vec{a}_2)$$

$$\vec{r}_1 \times \vec{F}_1 + \vec{r}_2 \times \vec{F}_2 = m_1 (\vec{r}_1 \times \vec{a}_1) + m_2 (\vec{r}_2 \times \vec{a}_2)$$

$$\vec{r}_1 \times \vec{F}_{21} + \vec{r}_2 \times \vec{F}_{12} = (\vec{r}_1 - \vec{r}_2) \times \vec{F}_{21} = 0 \,!$$

ker je $\vec{r}_1 - \vec{r}_2 \parallel \vec{F}_{12}, \vec{F}_{21}$ (glej sliko)

$$\sum_i \vec{M}_i = \sum_i \vec{r}_i \times \vec{F}_i = \sum_j m_j (\vec{r}_j \times \vec{a}_j)$$

Navori vseh zunanjih sil okoli iste osi se seštevajo!

Togo telo: vse točke telesa se vrtijo z istim kotnim pospeškom: $\vec{\alpha}_i = \vec{\alpha}$

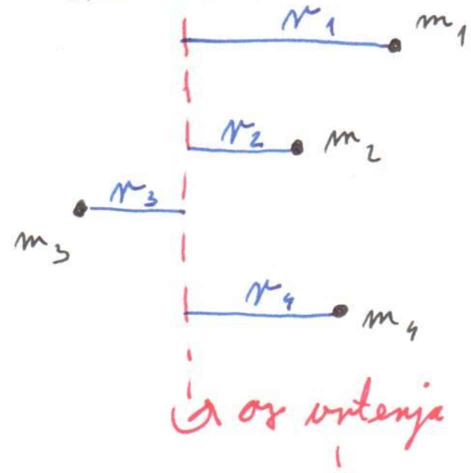
$$\vec{r}_j \times \vec{a}_j = r_j^2 \cdot \vec{\alpha}_j = r_j^2 \cdot \vec{\alpha} \,!$$

Vztrajnostni moment

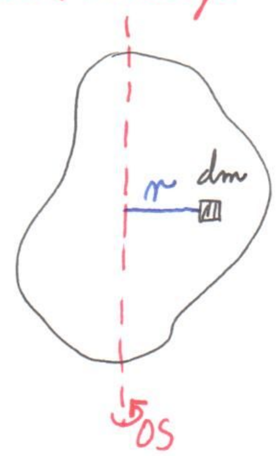
$$\boxed{\sum_i \vec{M}_i} = \sum_j m_j r_j^2 \cdot \vec{\alpha} = \boxed{J \cdot \vec{\alpha}}$$

– Vztrajnostni moment:

$$J = \sum_i r_i^2 m_i$$ za sistem točkastih teles, ki se vrti kot togo telo



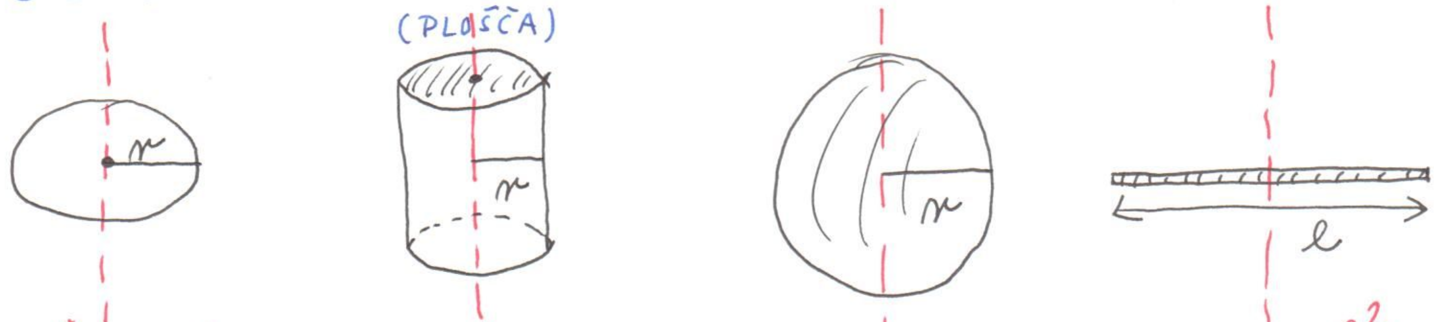
$$J = \int r^2 \, dm$$



J je odvisen od osi vrtenja!

J^*: vztrajnostni moment okoli težiščne osi

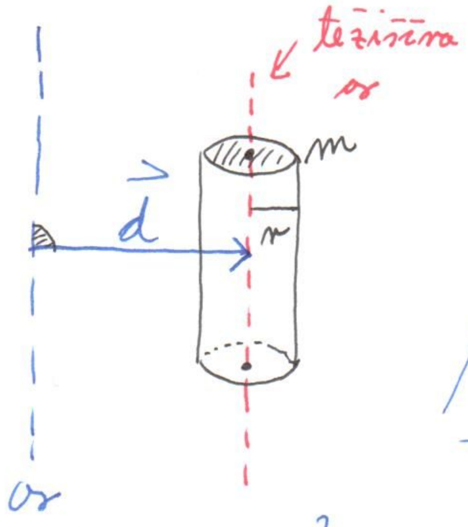
OBROČ — VALJ (PLOŠČA) — KROGLA — (TANKA) PALICA

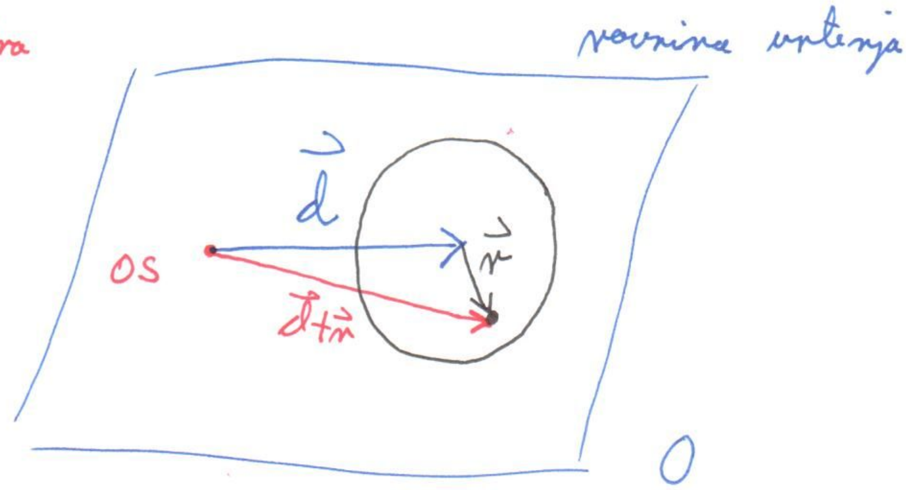


$J^* = m r^2$ — $J = \frac{1}{2} m r^2$ — $J = \frac{2}{5} m r^2$ — $J = \frac{m l^2}{12}$

Formule veljajo le za vrtenje okoli prikazanih (težiščnih) osi!

J okoli osi, ki je vzporedna s težiščno osjo:





$$J = \int (\vec{d}+\vec{r})^2\, dm = \int d^2\, dm + 2\vec{d}\cancel{\int \vec{r}\, dm} + \int r^2\, dm =$$

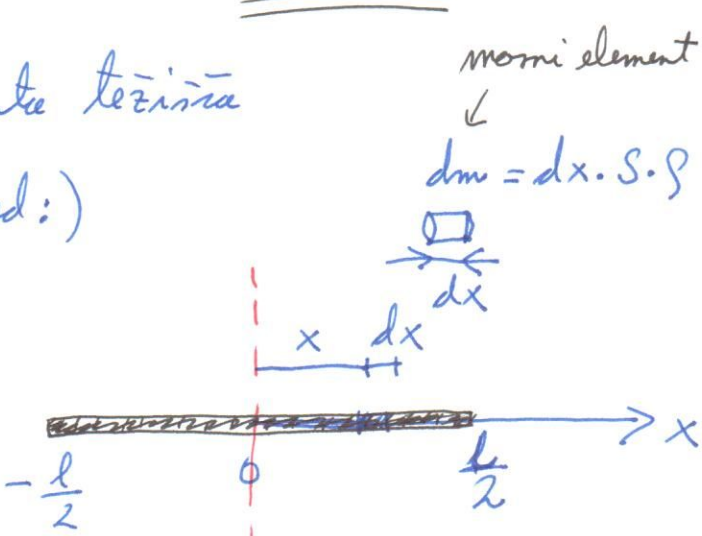
$$= d^2 \cdot m + J^*; \quad \boxed{J = d^2 m + J^*}$$

Steinerjev izrek!

d: razdalja med osjo vrtenja ter težiščno osjo. Pomembno: os vrtenja in težiščna os morata biti vzporedni!

$\vec{r}^{\,*} = \int \vec{r}\, dm / \int dm$: koordinate težišča

Vztrajnostni moment palice (zgled:)



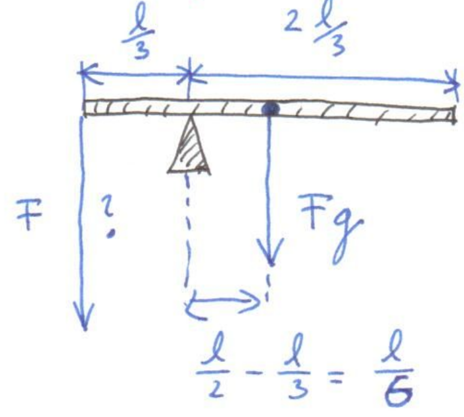
$$J = \int_{-\frac{l}{2}}^{\frac{l}{2}} x^2\, dm = \int_{-\frac{l}{2}}^{\frac{l}{2}} x^2 S\rho \cdot dx =$$

$$= S\rho \int_{-\frac{l}{2}}^{\frac{l}{2}} x^2\, dx = \frac{1}{3} S\rho\, x^3 \Big|_{-\frac{l}{2}}^{\frac{l}{2}} = \frac{1}{3}\, 2\, \frac{l^3}{8} S\rho = \frac{l^2}{12} \cdot \underbrace{l S\rho}_{m} = m\frac{l^2}{12}$$

Primeri uporabe navorov ter:

$$\sum_i \vec{M}_i = J \cdot \vec{\alpha}$$ Newtonov zakon za vrtenje

a.) $\sum_i \vec{M}_i = 0$ palica naj se ne vrti



$$\vec{r}_1 \times \vec{F} + \vec{r}_2 \times \vec{F}_g = 0$$

$$\frac{l}{3} \cdot F - \frac{l}{6} \cdot F_g = 0 \Rightarrow \boxed{F = \frac{F_g}{2}}$$

Sila F vrti v smeri, sila Fg pa $\Rightarrow$ − predznak.

Lahko tudi vektorsko: $\vec{r}_1 = \left(-\frac{l}{3}, 0, 0\right)$; $\vec{r}_2 = \left(\frac{l}{6}, 0, 0\right)$

$\vec{F} = (0, -F, 0)$; $\vec{F}_g = (0, -F_g, 0)$

$$\vec{r}_1 \times \vec{F} = \begin{vmatrix} i & j & k \\ -\frac{l}{3} & 0 & 0 \\ 0 & -F & 0 \end{vmatrix} = \left(0, 0, \frac{l}{3} F\right)$$

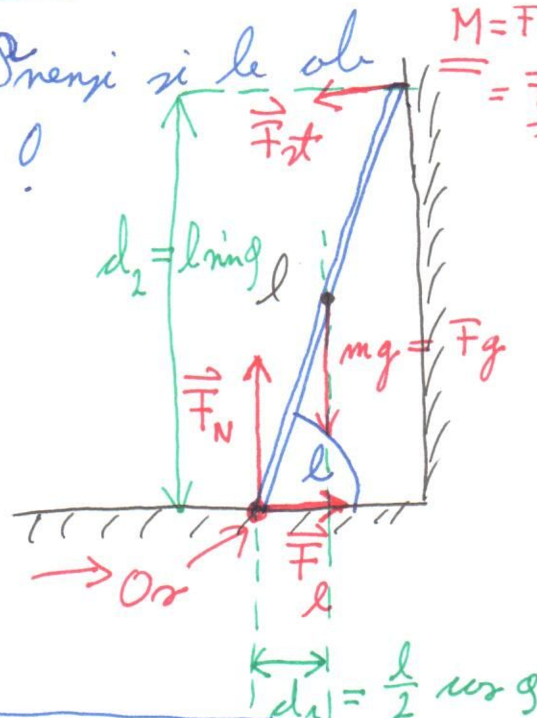
NEOBVEZNO

$$\vec{r}_2 \times \vec{F}_g = \left(0, 0, -\frac{l}{6} F_g\right)$$

---

b.) Izračunaj F_{st}: sila stene. Trenja ni le ob podlagi. $\sum_i \vec{F}_i = 0$ in $\sum_i \vec{M}_i = 0$!

$M = F r \sin\varphi = F \cdot d$



$\vec{F}_{st} + \vec{F}_l = 0$ vodoravne sile; $\vec{F}_N + \vec{F}_g = 0$ navpične sile

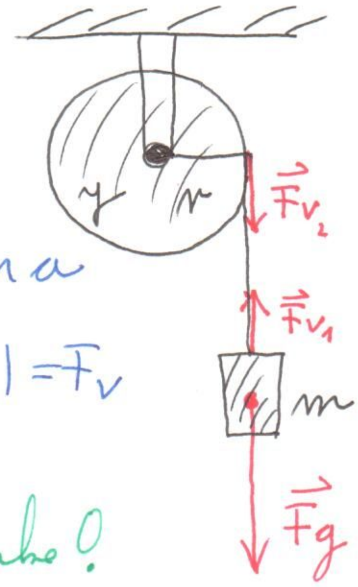
$\Rightarrow F_l = F_{st}$; $F_N = F_g$

$\sum_i \vec{M}_i = 0$ izberem os vrtenja na dnu $\rightarrow$ os

$$F_{st} \cdot d_2 - F_g d_1 = 0$$

$$F_{st} \cdot l\sin\varphi - mg\frac{l}{2}\cos\varphi = 0 \Rightarrow \boxed{F_{st} = \frac{mg}{2}\operatorname{ctg}\varphi}$$

c.) Vreteno:



Valj: $\vec{M} = J \cdot \vec{\alpha}$; $r \cdot F_V = J\alpha$

utež : $\vec{F}_g + \vec{F}_{V1} = m\vec{a}$; $mg - F_V = ma$

III Newt. zakon: $\vec{F}_{V1} + \vec{F}_{V2} = 0 \Rightarrow |\vec{F}_{V1}| = |\vec{F}_{V2}| = F_V$

Valj: $r\,F_V = J\,\alpha$ 3 neznanke!

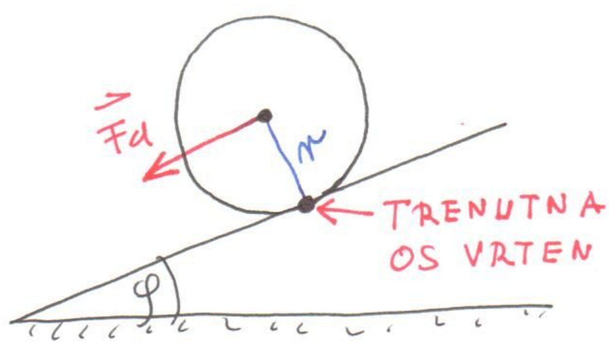
utež $mg - F_V = m\,a$ $\boxed{a = r \cdot \alpha}$; kajti $a = a_t$!

$$F_V = J\frac{\alpha}{r} = J\frac{a}{r^2}$$

$$mg - F_V = ma$$

$$mg = \left(\frac{J}{r^2} + m\right)a \Rightarrow \boxed{a = g\frac{m}{\left(m + \frac{J}{r^2}\right)}}$$

d.) Kotaljenje valja:



$r \cdot F_d = J \cdot \alpha$

Treba J okoli trenutne osi!

$J = J^* + mr^2 = \frac{mr^2}{2} + mr^2 = \frac{3}{2}mr^2$

$\alpha = \frac{a}{r}$; a: pospešek težišča!

$\cancel{r}\,\cancel{m}g\sin\varphi = \frac{3}{2}\cancel{m}r^{\cancel{2}}\frac{a}{\cancel{r}} \Rightarrow \boxed{a = \frac{2}{3}g}$

Izrek o vrtilni količini

$$\vec{M} = J\vec{\alpha}\ ; \quad \vec{\alpha} = \frac{d\vec{\omega}}{dt} \Rightarrow \vec{M} = J\frac{d\vec{\omega}}{dt}$$

$$\underbrace{\int_{t_z}^{t_k} \vec{M}\,dt}_{\text{sunek navora}} = \int_{\vec{\omega}_z}^{\vec{\omega}_k} J\,d\vec{\omega} = \underbrace{J\vec{\omega}_k - J\vec{\omega}_z}_{\text{sprememba vrtilne količine}}$$

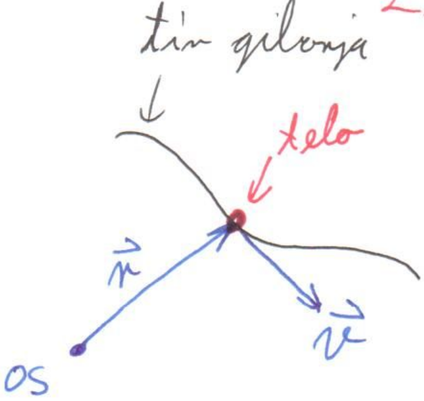
$$\int_{t_z}^{t_k} \vec{M}\,dt = J\vec{\omega}_k - J\vec{\omega}_z = \vec{\Gamma}_k - \vec{\Gamma}_z$$

Vrtilna količina: $\vec{\Gamma} = J\vec{\omega}$

Bolj splošno: $\int \vec{M}\,dt = J_k\omega_k - J_z\omega_z$

Med gibanjem se namreč lahko spremeni tudi vztrajnostni moment.

Zveza med gibalno $\vec{G}$ in vrtilno $\vec{\Gamma}$ količino:



$$\vec{F} = m\vec{a} \quad / \vec{r}\times$$

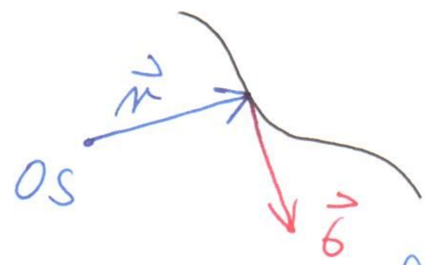
$$\vec{M} = \vec{r}\times\vec{F} = m\,\vec{r}\times\vec{a} = m\,\vec{r}\times\frac{d\vec{v}}{dt} =$$

$$= m\frac{d(\vec{r}\times\vec{v})}{dt} = \frac{d}{dt}(\vec{r}\times \underbrace{m\vec{v}}_{=\vec{G}}) = \frac{d}{dt}(\vec{r}\times\vec{G})$$

$$\vec{M} = \frac{d}{dt}\left(\vec{r}\times\underset{\substack{\uparrow\\ \text{gibalna količina}}}{\vec{G}}\right) = \frac{d}{dt}\underset{\substack{\uparrow\\ \text{vrtilna količina}}}{\vec{\Gamma}}$$

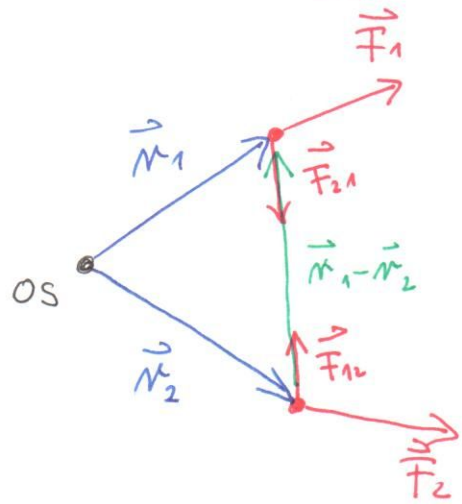
Velja tudi zveza $\vec{M} = \frac{d\vec{\Gamma}}{dt}$ ter

$$\vec{\Gamma} = \vec{r}\times\vec{G}$$



Vrtilno količino lahko pripišemo tudi telesu, ki se po nekem tiru giblje mimo izbrane osi. (čeprav se ne vrti!)

Izrek o vrtilni količini za sistem točkastih teles



$$\int \vec{r}_1\times\vec{F}_1\,dt + \int \vec{r}_1\times\vec{F}_{21}\,dt = \overbrace{m_1\vec{r}_1\times\vec{v}_{k1}}^{\vec{\Gamma}_{1k}} - m_1\vec{r}_1\times\vec{v}_{z1}$$

$$+$$

$$\int \vec{r}_2\times\vec{F}_2\,dt + \int \vec{r}_2\times\vec{F}_{12}\,dt = m_2\vec{r}_2\times\vec{v}_{k2} - m_2\vec{r}_2\times\vec{v}_{z2}$$

SEŠTEJEMO! $\quad = 0$

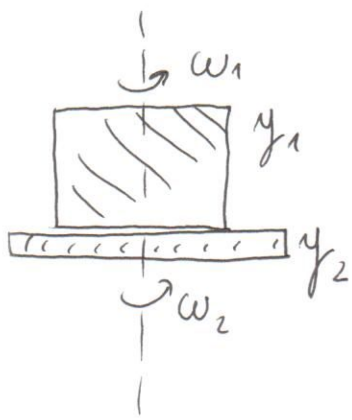
$$\int \vec{r}_1\times\vec{F}_1\,dt + \int \vec{r}_2\times\vec{F}_2\,dt = m_1\vec{r}_1\times\vec{v}_{k1} + m_2\vec{r}_2\times\vec{v}_{k2} - (m_1\vec{r}_1\times\vec{v}_{z1} + m_2\vec{r}_2\times\vec{v}_{z2})$$

$$\sum_{\substack{i\\ \text{po zun. navorih}}} \int \vec{M}_i\,dt = \sum_{\substack{j\\ \text{po vseh vrtilnih količinah}}} \vec{\Gamma}_{kj} - \vec{\Gamma}_{zj}$$

IZREK: Vsota sunkov navorov zunanjih sil okoli izbrane osi je enaka spremembi skupne vrtilne količine! POZOR: Vrtilne količine okoli skupne osi več teles se seštevajo! (a le okoli iste osi!)

Primeri:

a.) Ohranitev vrtilne količine: $\sum_i \int \vec{M}\,dt = 0 \Rightarrow \vec{\Gamma}_z = \vec{\Gamma}_k$



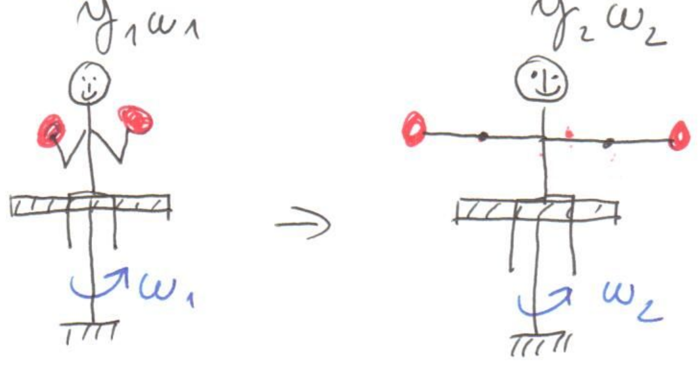
Dva valja se vrtita okoli skupne osi a na začetku z različnima ω_1, ω_2. Sunki navorov so 0! $\vec{\Gamma}$ se ohranja

$$\underbrace{J_1\omega_1 + J_2\omega_2}_{\vec{\Gamma}_z} = \underbrace{(J_1 + J_2)\,\omega_k}_{\vec{\Gamma}_k}$$

Smer $\vec{\Gamma}$ je smer osi vrtenja $\Rightarrow$

$$\boxed{\omega_k = \frac{J_1\omega_1 + J_2\omega_2}{J_1 + J_2}}$$

b.) Vrtenje na stolu:



$$J_1\omega_1 = J_2\omega_2$$

$$\Downarrow$$

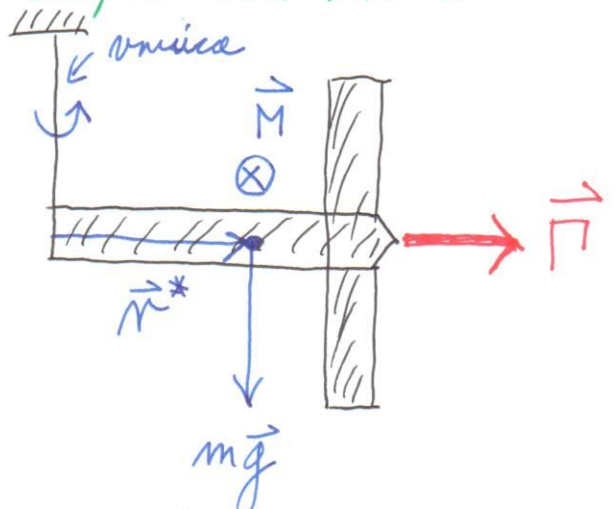
$$\frac{J_1}{J_2} = \frac{\omega_2}{\omega_1}$$

Z "odročenjem" spremenimo vztrajnostni moment $\Rightarrow$ spremeni se

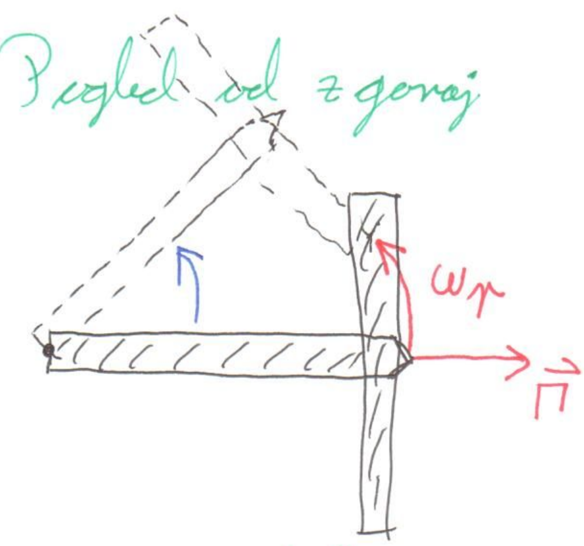
1.)

Vrtavka

Pogled od strani



Pogled od zgoraj



$\Rightarrow$

$$\vec{M} = \frac{d\vec{\Gamma}}{dt} = \vec{r} \times m\vec{g}$$

Smer navora $\perp$ na $\vec{\Gamma}$ ter na ravnino lista (stranski pogled)

$\Delta\vec{\Gamma} = \vec{M}\Delta t$

$$\Delta\Gamma = M\Delta t = \Delta\varphi \cdot \Gamma \Rightarrow$$

$$\frac{\Delta\varphi}{\Delta t} = \omega_p = \frac{M}{\Gamma} = \frac{r^* mg}{J \cdot \omega}$$

Vrtavka se vrti okoli vzdolžne osi z $\omega \Rightarrow \Gamma = J\omega$ ob enem pa se ~~z~~ s precesijsko ω_p vrti v vodoravni ravnini:

$$\omega_p = \frac{r^* mg}{J\omega}$$

Kinetična energija in delo pri vrtenju

$$M = J\alpha = J\frac{d\omega}{dt} \quad |\cdot d\varphi$$

$$M\,d\varphi = J\cdot\frac{d\omega\,d\varphi}{dt} = J\omega\,d\omega$$

$$A = \int_{\varphi_z}^{\varphi_k} M\,d\varphi = \int_{\omega_z}^{\omega_k} J\omega\,d\omega = J\frac{\omega_k^2}{2} - J\frac{\omega_z^2}{2}$$

Delo pri vrtenju: $A = \int_{\varphi_z}^{\varphi_k} M\,d\varphi$ ali $A = M\cdot\varphi$

če je M konstanten

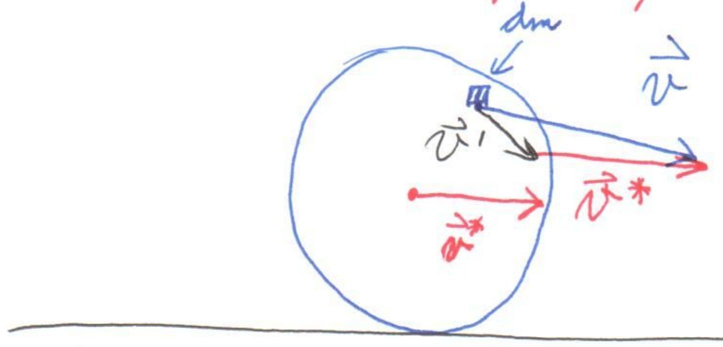
Rotacijska kinetična energija: $W_{rot} = J\frac{\omega^2}{2}$

Moč pri vrtenju: $dA = M\,d\varphi \quad |\frac{d}{dt}$

$$\boxed{P = M\omega} \qquad P = \frac{dA}{dt} = \frac{d(M\,d\varphi)}{dt} = M\cdot\frac{d\varphi}{dt} = \underline{\underline{M\omega}}$$

$M \neq M(t)$ za časovno konstanten navor!

Kinetična energija pri translacijskem in rotacijskem gibanju togega telesa



$$\vec v = \vec v^* + \vec v'$$

Celotna hitrost: $\vec v$; Hitrost težišča: $\vec v^*$

$\vec v'$: hitrost, merjena glede na koordinatni sistem, ki miruje glede na težišče telesa.

$$W_k = \frac{1}{2} \int_{\text{po telesu}} \vec v^{\,2}\, dm = \frac{1}{2} \int (\vec v^* + \vec v')^2\, dm =$$

$$= \frac{m v^{*2}}{2} + \frac{\vec v^*}{2} \underbrace{\cancel{\int \vec v'\, dm}}_{0} + \frac{1}{2} \int \underbrace{v'^2}_{r^2 \omega^2} dm =$$

$$= \frac{m v^{*2}}{2} + \frac{\omega^2}{2} \int_{\text{po telesu}} r^2\, dm = \frac{m v^{*2}}{2} + J \frac{\omega^2}{2}$$

v^*: hitrost težišča

ω : kotna hitrost vrtenja okoli težiščne osi!