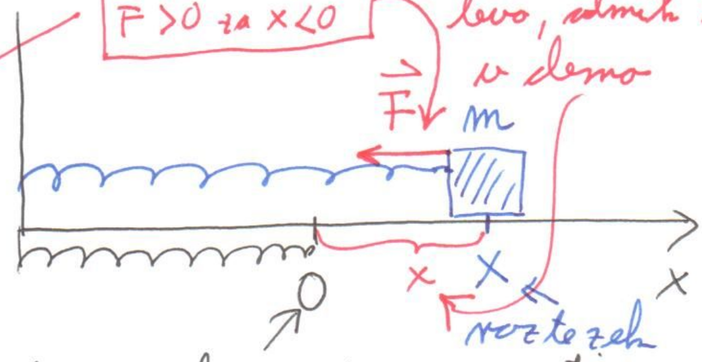
Sinusno nihanje

- Nihalo na vijačno vzmet

$F < 0$ za $x > 0$
$F > 0$ za $x < 0$

Sila kaže v levo, odmik x v desno



lega neobremenjene vzmeti

Newtonov zakon:

$$m\vec{a} = \vec{F} \quad ; \quad F = -kx$$

$$ma = -kx$$

kjer $a = \frac{dv}{dt} = \frac{d^2x}{dt^2}$

$$\Rightarrow m\frac{d^2x}{dt^2} = -kx \quad (*)$$

: Diferencialna enačba! Iščemo $x(t)$, ki za $\forall t$ reši enačbo.

Nastavek: $\boxed{x(t) = x_0 \sin \omega t}$

amplituda nihanja (x_0), krožna frekvenca (ω)

ali $x = x_0 \cos \omega t$!

ali $x = x_0 \sin(\omega t + \delta)$

$$\boxed{v = \frac{dx}{dt} = x_0 \omega \cos \omega t}$$

$$\boxed{a = \frac{d^2x}{dt^2} = -x_0 \omega^2 \sin \omega t}$$ } vstavimo v (*)

$$-m\underbrace{\cancel{x_0}\,\omega^2 \cancel{\sin \omega t}}_{-\frac{d^2x}{dt^2}} = -k\underbrace{\cancel{x_0} \cancel{\sin \omega t}}_{x}$$

$$m\omega^2 = k \Rightarrow \omega^2 = \frac{k}{m}$$

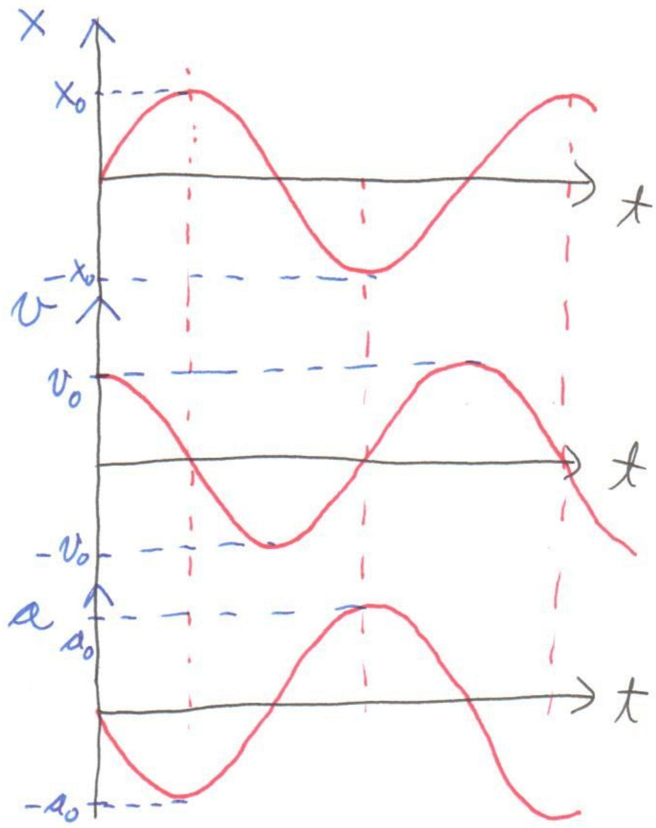
$$\omega = \sqrt{\frac{k}{m}} \quad ; \quad \omega = \frac{2\pi}{t_0} \Rightarrow t_0 = 2\pi\sqrt{\frac{m}{k}}$$

NIHAJNI ČAS!

$$x = x_0 \sin \omega t$$

$$v = \overbrace{x_0 \omega}^{v_0} \cos \omega t$$

$$a = -\overbrace{x_0 \omega^2}^{a_0} \sin \omega t$$



Energija pri sinusnem nihanju (nihala na vijačno vzmet)

$$W = W_{kin} + W_{pr}$$

Ima dva prispevka:

W_{kin}: kinetična energija

W_{pr}: prožnostna energija

$$W = m\frac{v^2}{2}{(*)} + k\frac{x^2}{2}{(*)}$$

Za v in x vstavimo gornja izraza

$$W = m \frac{x_0^2 \omega^2}{2} \cos^2 \omega t + k \frac{x_0^2}{2} \sin^2 \omega t$$

velja tudi $m\omega^2 = k$

$$W = \frac{k x_0^2}{2} \cos^2 \omega t + k \frac{x_0^2}{2} \sin^2 \omega t = \frac{k x_0^2}{2} \underbrace{\left(\cos^2 \omega t + \sin^2 \omega t\right)}_{1}$$

$$W = \frac{k x_0^2}{2} = m \frac{v_0^2}{2}$$

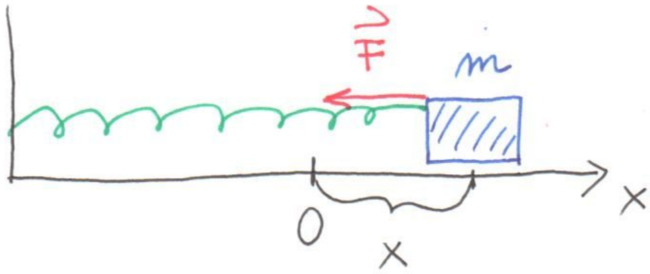
celotna energija NI odvisna od časa (je konstanta)!

Maksimalna W_{pr} Maksimalna W_{kin}

$$k \frac{x_0^2}{2} = m \frac{v_0^2}{2} = k \frac{x^2}{2} + m \frac{v^2}{2}$$

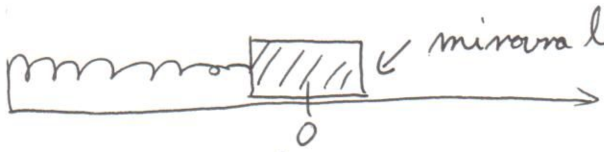
Sinusno nihanje

a.) Nihalo na vijačno vzmet



$F = -kx$

ker $F < 0$ za $x > 0$ in $F > 0$ za $x < 0$!



mirovna lega, vzmet ni raztegnjena, $x = 0$

Newtonov zakon: $-kx = ma \;;\; a = \dfrac{d^2x}{dt^2}$

$$-kx = m\frac{d^2x}{dt^2} \Leftarrow \text{diferencialna enačba za } x(t)\,!$$

Nastavek: amplituda, kotna hitrost (frekvenca)

$$x = x_0 \sin \omega t$$

$$v = \frac{dx}{dt} = x_0 \omega \cos \omega t$$

$$a = \frac{dv}{dt} = -x_0 \omega^2 \sin \omega t = \frac{d^2x}{dt^2}$$

$$-k\cancel{x_0 \sin\omega t} = -\cancel{x_0} m \omega^2 \cancel{\sin \omega t}$$

$$\omega^2 = \frac{k}{m} \Rightarrow \omega = \sqrt{\frac{k}{m}}$$

Nihajni čas: $\omega t_0 = 2\pi \Rightarrow t_0 = \dfrac{2\pi}{\omega}$

$$t_0 = 2\pi\sqrt{\frac{m}{k}}$$

$$\left.\begin{aligned} W_k &= \frac{mv^2}{2} = \frac{m}{2} x_0^2 \omega^2 \cos^2 \omega t \\ W_{pr} &= k\frac{x^2}{2} = \frac{k}{2} x_0^2 \sin^2 \omega t \end{aligned}\right\} W = W_k + W_{pr} \Rightarrow$$

Celotna energija se pri sinusnem nihanju ohranja:

$$W = \frac{x_0^2}{2} \underbrace{m\omega^2}_{=k} \cos^2 \omega t + \frac{x_0^2}{2} k \sin^2 \omega t =$$

$$= k \frac{x_0^2}{2} (\cos^2 \omega t + \sin^2 \omega t) = k \frac{x_0^2}{2} = m \frac{\overbrace{x_0^2 \omega^2}^{=v_0^2}}{2}$$

maksimalna W_{pr} maksimalna W_{kin}

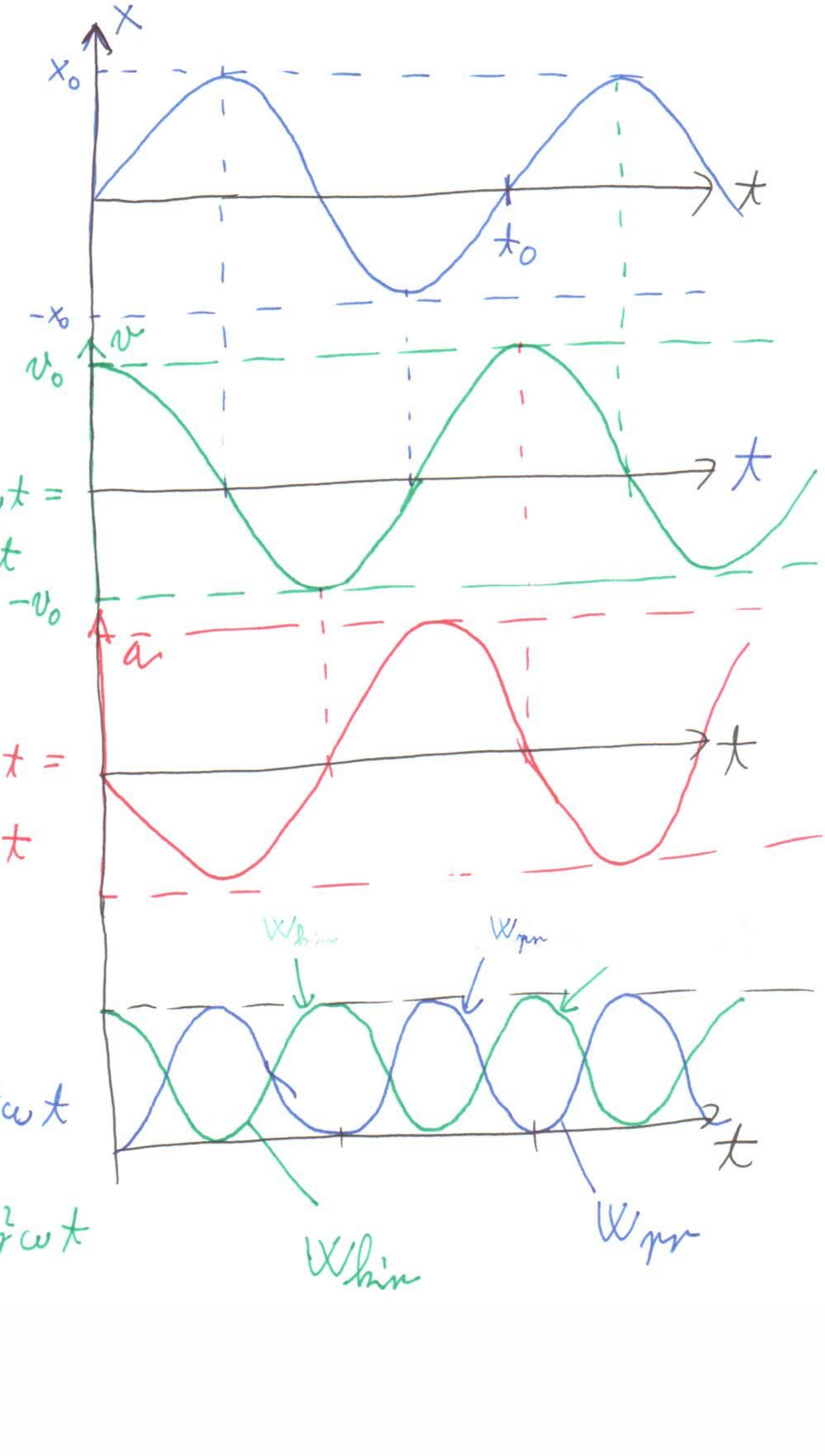
$x = x_0 \sin \omega t$

$v = x_0 \omega \cos \omega t =$
$= v_0 \cos \omega t$

$a = -x_0 \omega^2 \sin \omega t =$
$= -a_0 \sin \omega t$

$W_{pr} = \frac{k x_0^2}{2} \sin^2 \omega t$

$W_{kin} = \frac{m v_0^2}{2} \cos^2 \omega t$



Izpeljava enačbe za sinusno nihanje – malo drugače – iz ohranitve energije: (NEOBVEZNO!)

$$W = k\frac{x_0^2}{2} = k\frac{x^2}{2} + m\frac{v^2}{2}$$

$$m v^2 = k\,(x_0^2 - x^2)$$

$$v^2 = \frac{k}{m}(x_0^2 - x^2) = \omega^2 (x_0^2 - x^2)$$

$$v = \frac{dx}{dt} = \omega\sqrt{x_0^2 - x^2}$$

$$\Downarrow t$$

$$\int_{x_1}^{x} \frac{dx}{\sqrt{x_0^2 - x^2}} = \int_0^t \omega\, dt = \omega t$$

$y = \frac{x}{x_0}$; $y_1 = \frac{x_1}{x_0}$

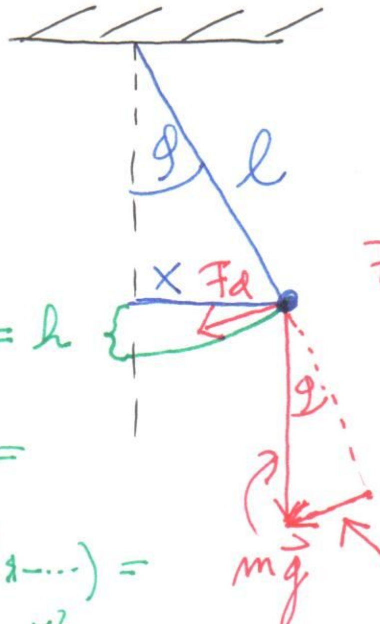
$$\int_{y_1}^{y} \frac{dy}{\sqrt{1-y^2}} = \arcsin y \Big|_{y_1}^{y} = \arcsin y - \arcsin y_1$$

φ faza je enaka $\arcsin \frac{x_1}{x_0}$, kjer je x_1 odmik ob času $t=0$

$$\arcsin \frac{x}{x_0} = \omega t + \overbrace{\arcsin \frac{x_1}{x_0}}^{\varphi}$$

$$\boxed{x = x_0 \sin(\omega t + \varphi)}$$

h.) Matematično nihalo:



$F_d = mg \sin\varphi$

$$m \frac{d^2x}{dt^2} = -F_d$$

$$m \frac{d^2x}{dt^2} = -mg \sin\varphi$$

$$\sin\varphi = \frac{x}{l}$$

$$m \frac{d^2x}{dt^2} = -mg \frac{x}{l}$$

$$l - l\cos\varphi = h$$

$$l(1-\cos\varphi) =$$

$$= l\left(1 - 1 + \frac{\varphi^2}{2} + \dots\right) =$$

$$= l\frac{\varphi^2}{2} = l\frac{x^2}{2l^2}$$

Nastavek: $x = x_0 \sin\omega t$

$$\frac{d^2x}{dt^2} = -x_0\omega^2 \sin\omega t$$

$$\frac{d^2x}{dt^2} = -\frac{g}{l} \cdot x$$

$$-x_0\omega^2 \sin\omega t = -\frac{g}{l} x_0 \sin\omega t$$

$$\omega = \sqrt{\frac{g}{l}}$$

Nihajni čas: $t_0 = 2\pi\sqrt{\frac{l}{g}}$

$$\varphi \sim \frac{x}{l}$$

$$\sin^2\frac{\varphi}{2} \sim \frac{\varphi^2}{4}$$

$$W_{kin} = \frac{mv^2}{2} = m\frac{x_0^2\omega^2}{2}\cos^2\omega t$$

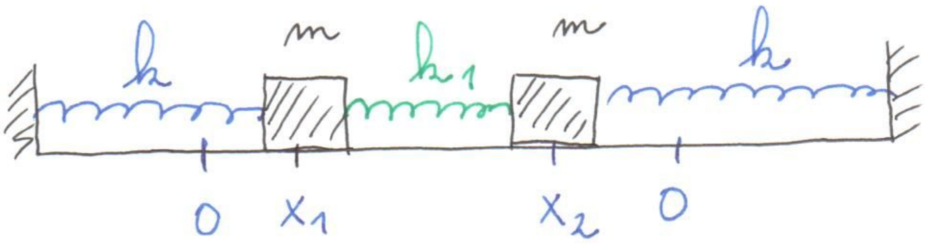
$$W_{pot} = mgh = mgl(1 - \overset{\cos^2\frac{\varphi}{2} - \sin^2\frac{\varphi}{2}}{\cos\varphi}) = 2mgl\sin^2\frac{\varphi}{2} =$$

$$= mgl\frac{\varphi^2}{2} = mg\frac{x^2}{2l} = mg\frac{x_0^2}{2l}\sin^2\omega t$$

$$W_{kin} + W_{pot} = m\frac{x_0^2\omega^2}{2} = m\frac{v_0^2}{2} = mg\frac{x_0^2}{2l}$$

Nihanje sestavljenega nihala

Poseben primer enakih nihal



x_1 in x_2 sta odmika prvega in drugega nihala od njunih ravnovesnih leg. V ravnovesnih legah so vse vzmeti neraztegnjene.

Enačbi gibanja za povezani nihali:

$$m\frac{d^2x_1}{dt^2} = -kx_1 - k_1\overbrace{(x_1 - x_2)}^{\text{raztezek vmesne vzmeti}}$$

$$m\frac{d^2x_2}{dt^2} = -kx_2 - k_1(x_2 - x_1)$$

novi spremenljivki: $y_1 = x_1 + x_2$ ter $y_2 = x_1 - x_2$

velja tudi za odvode: $\frac{d^2y_1}{dt^2} = \frac{d^2x_1}{dt^2} + \frac{d^2x_2}{dt^2}$

Enačbi

a.) Seštejemo: $m\frac{d^2y_1}{dt^2} = -ky_1$

b.) Odštejemo:

$$m\frac{d^2y_2}{dt^2} = -ky_2 - 2k_1y_2 = -(k+2k_1)y_2$$

Rešitve: $y_1 = y_{10}\sin\omega_1 t$; $y_2 = y_{20}\sin\omega_2 t$

$$\omega_1 = \sqrt{\frac{k}{m}} \; ; \quad \omega_2 = \sqrt{\frac{k+2k_1}{m}} \; ; \quad \omega_2 > \omega_1 \; !$$

Odmiki: $x_1 = \frac{1}{2}(y_1 + y_2)$

$x_2 = \frac{1}{2}(y_1 - y_2)$

$$x_1 = \frac{1}{2} y_{10} \sin\omega_1 t + \frac{1}{2} y_{20} \sin\omega_2 t$$

$$x_2 = \frac{1}{2} y_{10} \sin\omega_1 t - \frac{1}{2} y_{20} \sin\omega_2 t$$

Primer: lastni nihanji: $y_{10} > 0$ ter $y_{20} = 0$ Prvo l.n.

ali $y_{10} = 0$ ter $y_{20} > 0$ Drugo l.n.

$\Rightarrow \quad x_1 = x_2 = \frac{1}{2} y_{10} \sin \underline{\omega_1} t$: Prvo lastno nihanje

$x_1 = -x_2 = \frac{1}{2} y_{20} \sin \underline{\omega_2} t$: Drugo lastno nihanje

Energija nihanja:

$$W = m\frac{v_1^2}{2} + m\frac{v_2^2}{2} + k\frac{x_1^2}{2} + k\frac{x_2^2}{2} + k_1\frac{(x_1 - x_2)^2}{2}$$

Prvo l.n. $x_1 = x_2$ ter $v_1 = v_2 \Rightarrow x_1 = x_2 = x_0 \sin\omega_1 t$

$$W_1 = 2 m\frac{v_1^2}{2} + 2k\frac{x_1^2}{2} = m v_0^2 = k x_0^2 \; ; \quad v_0 = x_0 \omega$$

Drugo l.n. $\quad x_1 = -x_2 = x_0 \sin\omega_2 t$

$$W_2 = 2m\frac{v_1^2}{2} + 2k\frac{x_1^2}{2} + k_1\frac{(2x_1)^2}{2} = m v_1^2 + (k + 2k_1)x_1^2 =$$

$$= m\omega_0^2 x_0^2 = k x_0^2$$

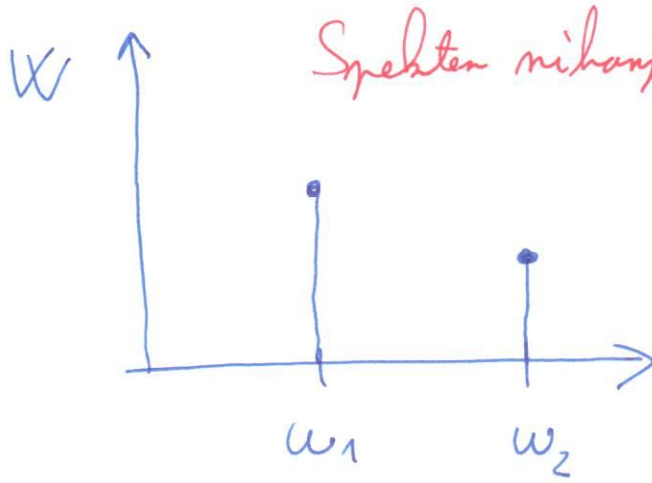
Splošno nihanje dveh sestavljenih nihal: (50)

$$X_1 = X_{I0} \sin(\omega_1 t + \varphi_1) + X_{II0} \sin(\omega_2 t + \varphi_2)$$

$$X_2 = X_{I0} \sin(\omega_1 t + \varphi_1) - X_{II0} \sin(\omega_2 t + \varphi_2)$$

$$\boxed{\begin{aligned} X_{I0} &= \tfrac{1}{2} Y_{10} \\ X_{II0} &= \tfrac{1}{2} Y_{20} \end{aligned}}$$

Spekter nihanja

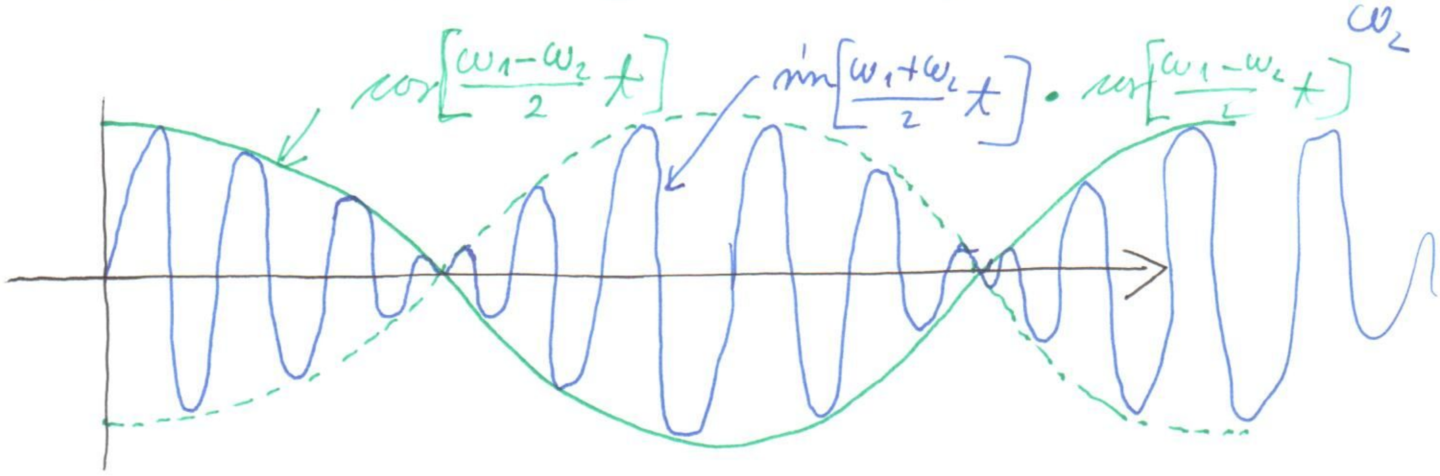


Utripanje: $X_{I0} = X_{II0}$; $\varphi_1 = \varphi_2 = 0$ ter $\omega_1 \sim \omega_2$ vendar $\omega_1 \neq \omega_2$!

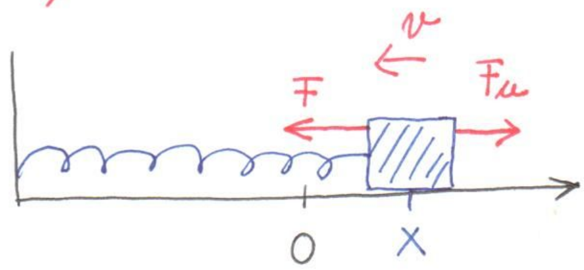
$$X_1 = X_{I0}(\sin\omega_1 t + \sin\omega_2 t) =$$

$$= 2 X_{I0} \sin\left[\frac{\omega_1+\omega_2}{2} t\right] \cos\left[\frac{\omega_1-\omega_2}{2} t\right]$$

Ker velja $\omega_1 \sim \omega_2 \Rightarrow \frac{\omega_1+\omega_2}{2} \sim \omega_1 \sim \omega_2$, torej $|\omega_1 - \omega_2| \ll \omega_1, \omega_2$



Dušeno nihanje



$$ma = -kx - 2m\beta v$$

$$m\frac{d^2x}{dt^2} = -kx - 2m\beta\frac{dx}{dt} \quad ; \quad (1)$$

Nastavek: $x = x_0 e^{-\alpha t} \sin\omega t$

$$\frac{dx}{dt} = -x_0\alpha e^{-\alpha t}\sin\omega t + x_0\omega e^{-\alpha t}\cos\omega t$$

$$\frac{d^2x}{dt^2} = x_0\alpha^2 e^{-\alpha t}\sin\omega t - 2x_0\alpha\omega e^{-\alpha t}\cos\omega t -$$

$$- x_0\omega^2 e^{-\alpha t}\sin\omega t$$

Vstavimo v enačbo (1):

$$x_0 m\alpha^2 e^{-\alpha t}\sin\omega t - 2x_0 m\alpha\omega e^{-\alpha t}\cos\omega t -$$

$$- x_0 m\omega^2 e^{-\alpha t}\sin\omega t = -k x_0 e^{-\alpha t}\sin\omega t + 2m\beta x_0\alpha e^{-\alpha t}\sin\omega t$$

$$- 2m\beta x_0\omega e^{-\alpha t}\cos\omega t$$

$$x_0 m\alpha^2 - x_0 m\omega^2 = -k x_0 + 2m\alpha\beta x_0$$

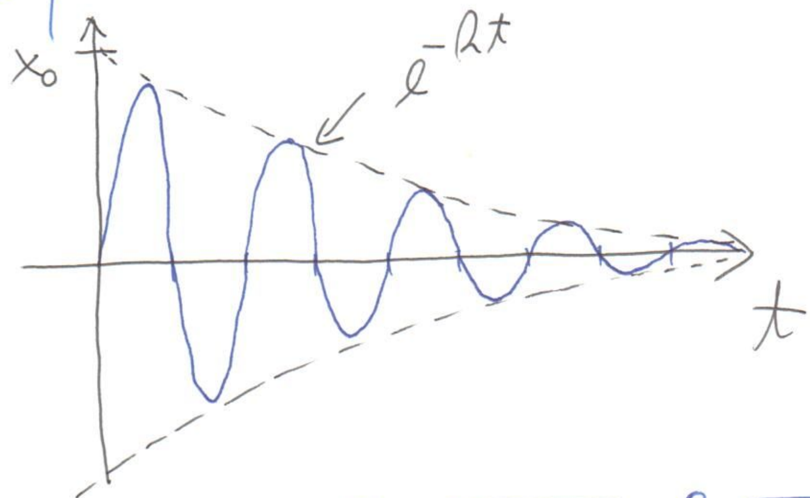
$$-\cancel{2x_0 m}\alpha\cancel{\omega} = -\cancel{2m}\cancel{x_0}\beta\cancel{\omega} \Rightarrow \alpha = \beta$$

$$\omega^2 = \frac{k}{m} - \beta^2 \Rightarrow \boxed{\omega = \sqrt{\frac{k}{m} - \beta^2} = \sqrt{\omega_0^2 - \beta^2}}$$

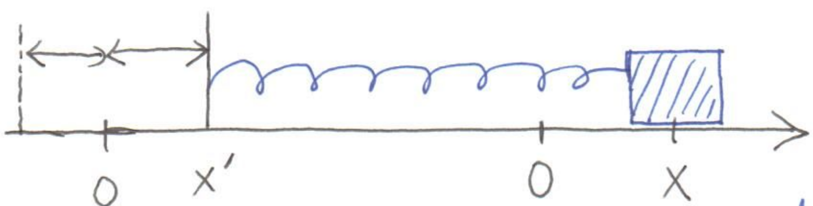
$$\omega_0^2 = \frac{k}{m}$$

Rešitev: $x = x_0\, e^{-\beta t} \sin \omega t$;

$\omega = \sqrt{\frac{k}{m} - \beta^2}$; $\sqrt{\frac{k}{m}} < \beta \Rightarrow$ Kritično dušenje



Vsiljeno nihanje



$\boxed{x' = x_0 \cos \underline{\omega} t}$ Vzbujanje!

$$m \overset{a}{\frac{d^2x}{dt^2}} = -k(x - x') - 2\beta m \overset{v}{\frac{dx}{dt}}$$

$$m \frac{d^2x}{dt^2} + 2m\beta \frac{dx}{dt} + kx = kx' \quad (1)$$

Nastavek

$$x = \underbrace{x_2\, e^{-\beta t} \sin(\omega' t - \delta')}_{\substack{\text{Dušeno nihanje z lastno} \\ \text{frekvenco dušenega nihala} \\ \text{I}}} + \underbrace{x_1 \cos(\underline{\omega} t - \delta)}_{\substack{\text{Nedušeno nihanje s} \\ \text{frekvenco vzbujanja!} \\ \text{II}}}$$

Po dolgem času $t \gg 1/\beta$ prevlada II člen.

II del nastavka vstavimo v enačbo (1)

$$-m\omega^2 x_1 \cos(\omega t-\vartheta) - 2m\beta\omega x_1 \sin(\omega t-\vartheta) +$$
$$+ k x_1 \cos(\omega t-\vartheta) = k x_0 \cos\omega t$$

$$\left.\begin{aligned}\cos(x+y) &= \cos x \cos y - \sin x \sin y\\ \sin(x+y) &= \sin x \cos y + \cos x \sin y\end{aligned}\right\}$$ adicijski izreki

$$-m\omega^2 x_1 (\underline{\cos\omega t}\cos\vartheta + \sin\omega t \sin\vartheta) - 2m\beta\omega x_1 \cdot$$
$$\cdot(\sin\omega t \cos\vartheta - \underline{\cos\omega t}\sin\vartheta) + k x_1(\underline{\cos\omega t}\cos\vartheta + \sin\omega t \sin\vartheta)$$
$$= k x_0 \underline{\cos\omega t}$$

$$(k x_1 - m\omega^2 x_1)\cos\vartheta + 2m\beta\omega x_1 \sin\vartheta = k x_0$$

$$-m\omega^2 \cancel{x_1} \sin\vartheta - 2m\beta\omega \cancel{x_1}\cos\vartheta + k\cancel{x_1}\sin\vartheta = 0$$

$$x = \frac{\omega}{\omega_0}$$

$$\operatorname{tg}\vartheta = \frac{2m\beta\omega}{k - m\omega^2} = \frac{2\beta\omega}{\omega_0^2 - \omega^2} = 2\frac{\beta}{\omega_0}\frac{x}{1-x^2};$$

$$x_1 = \frac{k x_0}{\cos\vartheta\,(k - m\omega^2 + 2m\beta\omega \operatorname{tg}\vartheta)} = \frac{\omega_0^2 x_0}{\cos\vartheta\,(\omega_0^2 - \omega^2 + 2\beta\omega\operatorname{tg}\vartheta)}$$

$$\cos^2\vartheta = \frac{1}{1+\operatorname{tg}^2\vartheta} = \frac{1}{1 + \frac{(2\frac{\beta}{\omega_0})^2 x^2}{(1-x^2)^2}}$$

$$x_1 = \frac{x_0}{\cos\vartheta\left(1-x^2+\left(\frac{2\beta}{\omega_0}\right)x\,\mathrm{tg}\,\vartheta\right)} =$$

$$= \frac{x_0}{\sqrt{\frac{1}{1+\frac{\left(2\frac{\beta}{\omega_0}\right)^2 x^2}{(1-x^2)^2}}}\left(1-x^2+\frac{\left(2\frac{\beta}{\omega_0}\right)^2 x^2}{1-x^2}\right)} =$$

$$= \frac{x_0}{\sqrt{(1-x^2)^2+\left(\frac{2\beta}{\omega_0}\right)^2 x^2}}$$