① a) $E_0 = -\frac{m\lambda^2}{2\hbar^2}$

1/4

$\psi_0(x) = \sqrt{\kappa}\, e^{-\kappa|x-x_0|}\ ;\ \kappa = \frac{m\lambda}{\hbar^2}$ } iz vaj

b) $\tilde{x} \equiv x - x_0$

1/4+ $\psi_0(\tilde{x}) = \sqrt{\kappa}\, e^{-\kappa|\tilde{x}|}$

$\langle \tilde{x} \rangle = 0 \quad (\psi_0(\tilde{x}) \text{ soda}) \Rightarrow \langle x \rangle = x_0$

$$\langle \tilde{x}^2 \rangle = \int_{-\infty}^{\infty} |\psi_0(\tilde{x})|^2 \tilde{x}^2 d\tilde{x} = 2\int_0^{\infty} \kappa e^{-2\kappa\tilde{x}} \tilde{x}^2 d\tilde{x} = \frac{2\kappa}{(2\kappa)^3}\int_0^{\infty} u^2 e^{-u} du =$$

$$= \frac{1}{2\kappa^2} \Rightarrow \delta^2(x) = \langle (x - \langle x \rangle)^2 \rangle = \langle (\tilde{x} - \langle \tilde{x} \rangle)^2 \rangle = \langle \tilde{x}^2 \rangle = \frac{1}{2\kappa^2}$$

$$\langle \tilde{p} \rangle = \int_{-\infty}^{\infty} \underbrace{\psi_0(\tilde{x})^*}_{\text{soda}} \underbrace{\left(-i\hbar \frac{d}{d\tilde{x}}\right) \psi_0(\tilde{x})}_{\text{liha}} d\tilde{x} = 0 \Rightarrow \langle p \rangle = \langle \tilde{p} \rangle = 0$$

$$\langle \tilde{p}^2 \rangle = \int_{-\infty}^{\infty} \psi_0(\tilde{x})^* \left(-\hbar^2 \frac{d^2}{d\tilde{x}^2} \psi_0(\tilde{x})\right) d\tilde{x} = \int_{-\infty}^{\infty} \psi_0(\tilde{x})^* (-\hbar^2) \Big[\kappa^2 \psi_0(\tilde{x}) -$$

$$- 2\kappa \psi_0(\tilde{x})\, \delta(\tilde{x})\Big] d\tilde{x} = -\hbar^2\kappa^2 + 2\hbar^2\kappa\, |\psi_0(0)|^2 = \hbar^2\kappa^2$$

$$\Rightarrow \delta^2 p = \langle (p - \langle p \rangle)^2 \rangle = \langle (\tilde{p} - \langle \tilde{p} \rangle)^2 \rangle = \langle \tilde{p}^2 \rangle = \hbar^2\kappa^2$$

c) • $x = \frac{\tilde{x}_0}{\sqrt{2}}(a + a^+)$ $\qquad \tilde{x}_0 = \sqrt{\frac{\hbar}{m\omega}}$

1/4+ iz vaj $\qquad \tilde{p}_0 = \frac{\hbar}{\tilde{x}_0}$

$$x(t) = \frac{x_0}{\sqrt{2}}\left(a(t) + a^+(t)\right) = \frac{\tilde{x}_0}{\sqrt{2}}\left(a e^{-i\omega t} + a^+ e^{i\omega t}\right) =$$

$$= \frac{\tilde{x}_0}{\sqrt{2}}\left[\frac{1}{\sqrt{2}}\left(\frac{x}{\tilde{x}_0} + i\frac{p}{\tilde{p}_0}\right)e^{-i\omega t} + \frac{1}{\sqrt{2}}\left(\frac{x}{\tilde{x}_0} - i\frac{p}{\tilde{p}_0}\right)e^{i\omega t}\right] =$$

$$= x\cos\omega t + p\frac{\tilde{x}_0}{\tilde{p}_0}\sin\omega t = x\cos\omega t + \frac{p}{m\omega}\sin\omega t$$

• $\dot{p}(t) = \frac{i}{\hbar}[H, p](t) = \frac{i}{\hbar}\left[\frac{1}{2}kx^2, p\right](t) = -k\, x(t)$

$\dot{x}(t) = \frac{i}{\hbar}[H, x](t) = \frac{i}{\hbar}\left[\frac{p^2}{2m}, x\right](t) = \frac{p(t)}{m}$

$\ddot{x}(t) = -\frac{k}{m}x(t) \Rightarrow x(t) = A\cos\omega t + B\sin\omega t$

$x(0) = x \Rightarrow A = x$

$\dot{x}(0) = \frac{p}{m} \Rightarrow B = \frac{p}{m\omega}$

d) $\langle x, t \rangle = \langle x \rangle \cos\omega t + \frac{\langle p \rangle}{m\omega}\sin\omega t = x_0\cos\omega t$

1/4+ $\langle x^2, t \rangle = \langle x^2 \rangle \cos^2\omega t + \frac{\langle xp + px \rangle}{m\omega}\cos\omega t \sin\omega t + \frac{\langle p^2 \rangle}{m^2\omega^2}\sin^2\omega t$

$\langle xp + px \rangle = -i\hbar \langle x\frac{d}{dx} + \frac{d}{dx}x \rangle \in i\mathbb{R}$, ker je $\psi_0(x)$ realna

$\langle xp + px \rangle \in \mathbb{R}$, ker je $xp + px$ hermitski } $\langle xp + px \rangle = 0$

$$\delta^2 x(t) = \langle x^2, t \rangle - \langle x, t \rangle^2 = \delta^2 x(0)\cos^2\omega t + \frac{\hbar^2\kappa^2}{m^2\omega^2}\sin^2\omega t =$$

$$= \delta^2 x(0)\left[\cos^2\omega t + \frac{2\hbar^2\kappa^4}{m^2\omega^2}\sin^2\omega t\right] = \delta^2 x(0)\left[\cos^2\omega t + 8\left(\frac{E_0}{\hbar\omega}\right)^2\sin^2\omega t\right]$$

② a) $E = \hbar\omega(n_x + n_y + 1)$ (iz vaj)

1/4 2. vzbujeno stanje: $n_x + n_y = 2 \Rightarrow E = 3\hbar\omega$, $3\times$ degenerirano;
baza: $|20\rangle, |11\rangle, |02\rangle$

1/4 b) $[H, L_z^2] = L_z[H, L_z] + [H, L_z]L_z = 0$ saj je $[H, L_z] = 0$ (iz vaj)

1/2 c) • $L_z^2\left[\alpha|20\rangle + \beta|11\rangle + \gamma|02\rangle\right] = \lambda\left[\alpha|20\rangle + \beta|11\rangle + \gamma|02\rangle\right]$

$L_z = -i\hbar(a_x^+ a_y - a_x a_y^+)$ (iz vaj)

$$\left.\begin{aligned} L_z|20\rangle &= i\hbar\sqrt{2}|11\rangle \\ L_z|11\rangle &= -i\hbar\sqrt{2}|20\rangle + i\hbar\sqrt{2}|02\rangle \\ L_z|02\rangle &= -i\hbar\sqrt{2}|11\rangle \end{aligned}\right\} \quad \begin{aligned} L_z^2|20\rangle &= 2\hbar^2|20\rangle - 2\hbar^2|02\rangle \\ L_z^2|11\rangle &= 4\hbar^2|11\rangle \Rightarrow |11\rangle \text{ je lastno stanje } L_z^2 \; (m=2) \\ L_z^2|02\rangle &= 2\hbar^2|02\rangle - 2\hbar^2|20\rangle \end{aligned}$$

v podprostoru $|20\rangle, |02\rangle$:

$$\begin{bmatrix} 2\hbar^2, & -2\hbar^2 \\ -2\hbar^2, & 2\hbar^2 \end{bmatrix}\begin{bmatrix}\alpha \\ \gamma\end{bmatrix} = \lambda\begin{bmatrix}\alpha \\ \gamma\end{bmatrix} \qquad (2\hbar^2 - \lambda)^2 - 4\hbar^4 = 0$$

$$\lambda = 0, 4\hbar^2$$

$\lambda = (m\hbar)^2$

$\lambda = 0$: $\dfrac{|20\rangle + |02\rangle}{\sqrt{2}}$ $(m=0)$

$\lambda = 4\hbar^2$: $\dfrac{|20\rangle - |02\rangle}{\sqrt{2}}$ $(m=2)$

• $\psi_0(x) = \dfrac{1}{\sqrt[4]{\pi x_0^2}} e^{-\frac{x^2}{2x_0^2}}$

$\psi_1(x) = \dfrac{\sqrt{2}x}{x_0}\psi_0(x)$

$\psi_2(x) = \dfrac{a^+\psi_1(x)}{\sqrt{2}} = \dfrac{1}{\sqrt{2}}\cdot\dfrac{1}{\sqrt{2}}\left(\dfrac{x}{x_0} - i\dfrac{p}{p_0}\right)\psi_1(x) = \dfrac{2\left(\frac{x}{x_0}\right)^2 - 1}{\sqrt{2}}\psi_0(x)$

$\psi_{20}(x,y) = \psi_2(x)\psi_0(y) = \dfrac{1}{\sqrt{2}}\left(2\left(\dfrac{x}{x_0}\right)^2 - 1\right)\psi_0(x)\psi_0(y) = \dfrac{1}{\sqrt{2}}\left(2\dfrac{r^2\cos^2\varphi}{x_0^2} - 1\right)f(r)$

$\psi_{02}(x,y) = \dfrac{1}{\sqrt{2}}\left(2\dfrac{r^2\sin^2\varphi}{x_0^2} - 1\right)f(r)$

$\psi_{11}(x,y) = \psi_1(x)\psi_1(y) = \dfrac{2xy}{x_0^2}\psi_0(x)\psi_0(y) = \dfrac{2r^2\cos\varphi\sin\varphi}{x_0^2}f(r)$

$L_z^2 = \left(-i\hbar\dfrac{\partial}{\partial\varphi}\right)^2 = -\hbar^2\dfrac{\partial^2}{\partial\varphi^2} \Rightarrow L_z^2(A\cos m\varphi + B\sin m\varphi) = \hbar^2 m^2[\ldots]$

$m = 0, 1, 2, \ldots$

$\psi_{11}(r,\varphi) \propto \sin 2\varphi$ je lastno stanje L_z^2 z $m=2$

$\dfrac{\psi_{20}(r,\varphi) + \psi_{02}(r,\varphi)}{\sqrt{2}} = \left(\dfrac{r^2}{x_0^2} - 1\right)f(r)$ je lastno stanje L_z^2 z $m=0$

$\dfrac{\psi_{20}(r,\varphi) - \psi_{02}(r,\varphi)}{\sqrt{2}} \propto \cos^2\varphi - \sin^2\varphi = \cos 2\varphi$ je lastno stanje L_z^2 z $m=2$

d) $(H + \gamma L_z^2)|m\rangle = (3\hbar\omega + \gamma\hbar^2 m^2)|m\rangle$

1/4 $|20\rangle = \dfrac{1}{\sqrt{2}}\left[\dfrac{|20\rangle + |02\rangle}{\sqrt{2}} + \dfrac{|20\rangle - |02\rangle}{\sqrt{2}}\right] \Rightarrow |20, t\rangle = e^{-3i\omega t}\cdot\dfrac{1}{\sqrt{2}}\left[\dfrac{|20\rangle + |02\rangle}{\sqrt{2}} + \dfrac{|20\rangle - |02\rangle}{\sqrt{2}}e^{-4i\gamma\hbar t}\right]$

$t_0 = \dfrac{\pi}{4\gamma\hbar} \Rightarrow |20, t_0\rangle = e^{-3i\omega t_0}|02\rangle$