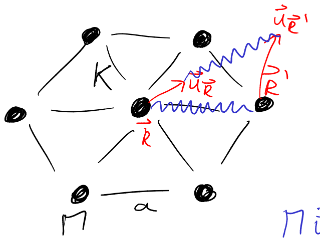


odmiki v ravnini kristalne mreže (v 2D)
(v 3D so odmiki v poljubni smeri)
vzmeti niso prednapete

$$M\ddot{\vec u}_{\vec R} = \sum_{\text{na } \vec R} \vec F$$

Lagrange: $L = T - V \longrightarrow \frac{\partial L}{\partial \vec u_{\vec R}} - \frac{d}{dt}\frac{\partial L}{\partial \dot{\vec u}_{\vec R}} = 0$

$$T = \sum_{\vec R} \frac{1}{2} M \left(\dot{\vec u}_{\vec R}\right)^2$$

$$V = \frac{1}{2}\sum_{\vec R' \neq \vec R} \frac{1}{2} K_{\vec R'-\vec R}\left[\underbrace{\left|\vec R' + \vec u_{\vec R'} - (\vec R + \vec u_{\vec R})\right|} - \cancel{|\vec R' - \vec R|}\right]^2$$

$$\frac{d}{dt}\frac{\partial L}{\partial \dot{\vec u}_{\vec R_0}} = \frac{d}{dt}\frac{\partial T}{\partial \dot{\vec u}_{\vec R_0}} = \frac{d}{dt} M\dot{\vec u}_{\vec R_0} = \boxed{M\ddot{\vec u}_{\vec R_0}}$$

$$\left|\vec R' - \vec R + (\vec u_{\vec R'} - \vec u_{\vec R})\right| = \sqrt{(\vec R' - \vec R + \vec u_{\vec R'} - \vec u_{\vec R})^2} =$$
$$= \sqrt{(\vec R' - \vec R)^2 + 2(\vec R' - \vec R)\cdot(\vec u_{\vec R'} - \vec u_{\vec R}) + \dots} =$$
$$= |\vec R' - \vec R|\sqrt{1 + 2\frac{(\vec R' - \vec R)\cdot(\vec u_{\vec R'} - \vec u_{\vec R})}{|\vec R' - \vec R|^2} + \dots} \doteq$$
$$\doteq |\vec R' - \vec R|\left(1 + \frac{(\vec R' - \vec R)\cdot(\vec u_{\vec R'} - \vec u_{\vec R})}{|\vec R' - \vec R|^2}\right) =$$
$$= \cancel{|\vec R' - \vec R|} + \frac{(\vec R' - \vec R)}{|\vec R' - \vec R|}\cdot(\vec u_{\vec R'} - \vec u_{\vec R}), \qquad \frac{(\vec R' - \vec R)}{|\vec R' - \vec R|} = \hat e_{\vec R' - \vec R}$$

$$V = \frac{1}{4}\sum_{\vec R' \neq \vec R} K_{\vec R' - \vec R}\left[(\vec u_{\vec R'} - \vec u_{\vec R})\cdot \hat e_{\vec R' - \vec R}\right]^2$$

$\forall \vec R_0$:

$$\frac{\partial \vec u_{\vec R'}}{\partial \vec u_{\vec R_0}} = \delta_{\vec R', \vec R_0}$$

$$\boxed{\frac{\partial L}{\partial \vec u_{\vec R_0}}} = -\frac{\partial V}{\partial \vec u_{\vec R_0}} = -\frac{1}{4}\sum_{\vec R' \neq \vec R} K_{\vec R' - \vec R}\, 2\left[(\vec u_{\vec R'} - \vec u_{\vec R})\cdot \hat e_{\vec R' - \vec R}\right]\left(\delta_{\vec R', \vec R_0} - \delta_{\vec R, \vec R_0}\right)\cdot \hat e_{\vec R' - \vec R} =$$

$$= -\frac{1}{2}\sum_{\vec R \neq \vec R_0} K_{\vec R_0 - \vec R}\left[(\vec u_{\vec R_0} - \vec u_{\vec R})\cdot \hat e_{\vec R_0 - \vec R}\right]\hat e_{\vec R_0 - \vec R} + \frac{1}{2}\sum_{\vec R \neq \vec R_0} \underbrace{K_{\vec R - \vec R_0}}_{= K_{\vec R_0 - \vec R}}\left[\underbrace{(\vec u_{\vec R} - \vec u_{\vec R_0})}_{= -(\vec u_{\vec R_0} - \vec u_{\vec R})}\cdot \underbrace{\hat e_{\vec R - \vec R_0}}_{= -\hat e_{\vec R_0 - \vec R}}\right]\underbrace{\hat e_{\vec R - \vec R_0}}_{= -\hat e_{\vec R_0 - \vec R}} =$$

$$= \boxed{-\sum_{\vec R \neq \vec R_0} K_{\vec R_0 - \vec R}\left[(\vec u_{\vec R_0} - \vec u_{\vec R})\cdot \hat e_{\vec R_0 - \vec R}\right]\hat e_{\vec R_0 - \vec R}}$$

$$-M\ddot{\vec u}_{\vec R_0} + \sum_{\vec R \neq \vec R_0} K_{\vec R - \vec R_0}\left[(\vec u_{\vec R} - \vec u_{\vec R_0})\cdot \hat e_{\vec R - \vec R_0}\right]\hat e_{\vec R - \vec R_0} = 0$$

nastavek: $\vec u_{\vec R} = \vec u_0 e^{i(\vec q \cdot \vec R - \omega t)}$

$$M\omega^2 \vec u_0 e^{i(\vec q\cdot \vec R_0 - \omega t)} + \sum_{\vec R \neq \vec R_0} K_{\vec R - \vec R_0}\left[e^{i(\vec q\cdot\vec R - \omega t)} - e^{i(\vec q\cdot \vec R_0 - \omega t)}\right](\vec u_0 \cdot \hat e_{\vec R - \vec R_0})\hat e_{\vec R - \vec R_0} = 0 \quad / \cdot e^{-i(\vec q\cdot\vec R_0 - \omega t)}$$

$$M\omega^2 \vec u_0 + \sum_{\vec R \neq \vec R_0} K_{\vec R - \vec R_0}\left[e^{i\vec q\cdot(\vec R - \vec R_0)} - 1\right](\vec u_0 \cdot \hat e_{\vec R - \vec R_0})\hat e_{\vec R - \vec R_0} = 0 \quad / \vec R_0 \to 0$$

$$M\omega^2 \vec u_0 + \sum_{\vec R \neq 0} K_{\vec R}\left[e^{i\vec q\cdot\vec R} - 1\right](\vec u_0\cdot\hat e_{\vec R})\hat e_{\vec R} = 0$$

$$M\omega^2 \vec u_0 + \frac{1}{2}\sum_{\vec R \neq 0} K_{\vec R}\left[e^{+i\vec q\cdot\vec R} - 1\right](\vec u_0\cdot\hat e_{\vec R})\hat e_{\vec R} + \frac{1}{2}\sum_{\vec R \neq 0} \underbrace{K_{-\vec R}}_{= K_{\vec R}}\left[e^{-i\vec q\cdot\vec R} - 1\right](\vec u_0\cdot\underbrace{\hat e_{-\vec R}}_{= -\hat e_{\vec R}})\underbrace{\hat e_{-\vec R}}_{= -\hat e_{\vec R}} = 0$$

$$M\omega^2 \vec u_0 + \frac{1}{2}\sum_{\vec R \neq 0} K_{\vec R}\Big(\underbrace{e^{i\vec q\cdot\vec R} + e^{-i\vec q\cdot\vec R} - 2}_{2\cos \vec q\cdot\vec R - 2 \,=\, -4\sin^2\frac{\vec q\cdot\vec R}{2}}\Big)(\vec u_0\cdot\hat e_{\vec R})\hat e_{\vec R}$$

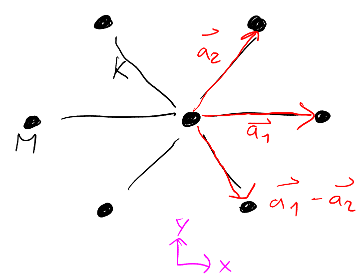
$$M\omega^2 \vec u_0 - 2\sum_{\vec R \neq 0} K_{\vec R}\sin^2\frac{\vec q\cdot\vec R}{2}\,(\vec u_0\cdot\hat e_{\vec R})\hat e_{\vec R} = 0$$

$$\left[(\vec a\cdot\vec b)\,\vec c\right]_\alpha = \left[(a_\beta b_\beta)\,\vec c\right]_\alpha = a_\beta b_\beta c_\alpha = \underbrace{c_\alpha b_\beta}_{(\vec c\otimes\vec b)_{\alpha\beta}} a_\beta = \left[(\vec c\otimes\vec b)\,\vec a\right]_\alpha$$

$$(\vec a\cdot\vec b)\,\vec c = (\vec c\otimes\vec b)\,\vec a$$

$$M\omega^2\vec u_0 - 2\sum_{\vec R\neq 0} K_{\vec R}\sin^2\frac{\vec q\cdot\vec R}{2}\left(\hat e_{\vec R}\otimes\hat e_{\vec R}\right)\vec u_0 = 0$$

$$\underbrace{\left[2\sum_{\vec R\neq 0} K_{\vec R}\sin^2\frac{\vec q\cdot\vec R}{2}\left(\hat e_{\vec R}\otimes\hat e_{\vec R}\right)\right]}_{d\times d \text{ matrika ; } d \text{ dimenzija kristala}}\vec u_0 = M\omega^2\vec u_0$$



$\vec a_1 = a(1,0)$ $\qquad -\vec a_1$

$\vec a_2 = a\left(\frac12, \frac{\sqrt3}{2}\right)$ $\qquad -\vec a_2$

$\vec a_1 - \vec a_2 = a\left(\frac12, -\frac{\sqrt3}{2}\right)$ $\qquad -(\vec a_1 - \vec a_2)$

$\hat e_{\vec a_1} = (1,0)$ $\qquad \hat e_{-\vec a_1} = -\hat e_{\vec a_1}$

$\hat e_{\vec a_2} = \left(\frac12, \frac{\sqrt3}{2}\right)$ $\qquad$ – ॥ –

$\hat e_{\vec a_1 - \vec a_2} = \left(\frac12, -\frac{\sqrt3}{2}\right)$ $\qquad$ – ॥ –

$(\vec a\otimes\vec b)_{ij} = a_i b_j$

$\hat e_{\vec a_1}\otimes\hat e_{\vec a_1} = \begin{pmatrix}1 & 0\\ 0 & 0\end{pmatrix}$ $\qquad \hat e_{-\vec a_1}\otimes\hat e_{-\vec a_1} = \hat e_{\vec a_1}\otimes\hat e_{\vec a_1}$

$\hat e_{\vec a_2}\otimes\hat e_{\vec a_2} = \begin{pmatrix}1/4, & \sqrt3/4\\ \sqrt3/4, & 3/4\end{pmatrix}$ $\qquad$ – ॥ –

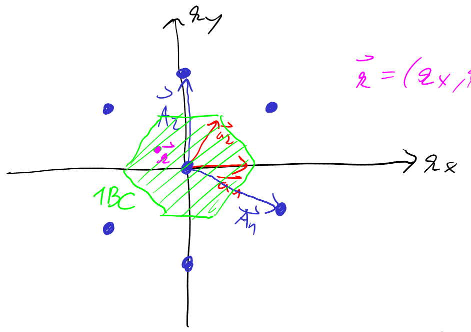
$\hat e_{\vec a_1-\vec a_2}\otimes\hat e_{\vec a_1-\vec a_2} = \begin{pmatrix}1/4, & -\sqrt3/4\\ -\sqrt3/4, & 3/4\end{pmatrix}$ $\qquad$ – ॥ –

$$\left(2\sum_{\vec R\neq 0} K_{\vec R}\sin^2\frac{\vec q\cdot\vec R}{2}\,\hat e_{\vec R}\otimes\hat e_{\vec R}\right)\vec u_0 = M\omega^2\vec u_0$$

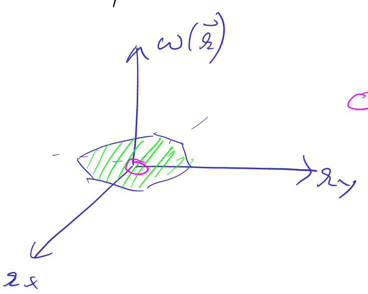
$$2K\left(\sin^2\frac{\vec q\cdot\vec a_1}{2}\,\hat e_{\vec a_1}\otimes\hat e_{\vec a_1} + \sin^2\frac{\vec q\cdot\vec a_2}{2}\,\hat e_{\vec a_2}\otimes\hat e_{\vec a_2} + \sin^2\frac{\vec q\cdot(\vec a_1-\vec a_2)}{2}\,\hat e_{\vec a_1-\vec a_2}\otimes\hat e_{\vec a_1-\vec a_2} + \right.$$
$$\left. + \sin^2\frac{\vec q\cdot(-\vec a_1)}{2}\,\hat e_{-\vec a_1}\otimes\hat e_{-\vec a_1} + \;\cdots\right)\vec u_0 = M\omega^2\vec u_0$$

$$4K\left[\sin^2\frac{q_x a}{2}\begin{pmatrix}1 & 0\\ 0 & 0\end{pmatrix} + \sin^2\frac{q_x+\sqrt3 k_y}{4}a\begin{pmatrix}1/4 & \sqrt3/4\\ \sqrt3/4 & 3/4\end{pmatrix} + \sin^2\frac{q_x-\sqrt3 k_y}{4}a\begin{pmatrix}1/4 & -\sqrt3/4\\ -\sqrt3/4 & 3/4\end{pmatrix}\right]\vec u_0 = M\omega^2\vec u_0$$

$$\begin{bmatrix}\sin^2\frac{q_x a}{2} + \frac14\left(\sin^2\frac{q_x+\sqrt3 q_y}{4}a + \sin^2\frac{q_x-\sqrt3 q_y}{4}a\right), & \frac{\sqrt3}{4}\left(\sin^2\frac{q_x+\sqrt3 q_y}{4}a - \sin^2\frac{q_x-\sqrt3 q_y}{4}a\right)\\ \frac{\sqrt3}{4}\left(\sin^2\frac{q_x+\sqrt3 q_y}{4}a - \sin^2\frac{q_x-\sqrt3 q_y}{4}a\right), & \frac34\left(\sin^2\frac{q_x+\sqrt3 q_y}{4}a + \sin^2\frac{q_x-\sqrt3 q_y}{4}a\right)\end{bmatrix}\vec u_0 = \frac{M\omega^2}{4K}\vec u_0$$



$\vec q = (q_x, q_y)$



$0 < |\vec q| \ll \frac{\pi}{a}$

dolgo valovna limita

$$\rightarrow \begin{bmatrix}\frac{q_x^2 a^2}{4} + \frac14\left(\left(\frac{q_x+\sqrt3 q_y}{4}a\right)^2 + \left(\frac{q_x-\sqrt3 q_y}{4}a\right)^2\right), & \frac{\sqrt3}{4}\left(\left(\;\right)^2 - \left(\;\right)^2\right)\\ \frac{\sqrt3}{4}\left(\left(\;\right)^2 - \left(\;\right)^2\right), & \frac34\left(\left(\;\right)^2 + \left(\;\right)^2\right)\end{bmatrix}\vec u_0 = \frac{M\omega^2}{4K}\vec u_0$$

$$\begin{bmatrix} \frac{q_x^2 a^2}{4} + \frac{1}{4}\left(2\left[\left(\frac{q_x}{4}\right)^2 + \left(\frac{\sqrt{3}k_y}{4}\right)^2\right] a^2\right) , & \frac{\sqrt{3}}{4} \cdot 2 \cdot 2 \frac{q_x}{4}\frac{\sqrt{3}q_y}{4} \cdot a^2 \\ \frac{\sqrt{3}}{4} 2 \cdot 2 \frac{q_x}{4}\frac{\sqrt{3}q_y}{4} a^2 , & \frac{3}{4}\left(2\left[\left(\frac{q_x}{4}\right)^2 + \left(\frac{\sqrt{3}k_y}{4}\right)^2\right] a^2\right) \end{bmatrix} \vec{u}_0 = \frac{M\omega^2}{4K}\vec{u}_0$$

$$\begin{bmatrix} \frac{q_x^2 a^2}{4} + \frac{q_x^2 a^2}{32} + \frac{3q_y^2 a^2}{32} , & \frac{3}{16} q_x q_y a^2 \\ \frac{3}{16} q_x q_y a^2 , & \frac{3}{32} q_x^2 a^2 + \frac{9}{32} q_y^2 a^2 \end{bmatrix} \vec{u}_0 = \frac{M\omega^2}{4K}\vec{u}_0$$

$$\begin{bmatrix} \frac{9q_x^2 a^2}{32} + \frac{3q_y^2 a^2}{32} , & \frac{6}{32} q_x q_y a^2 \\ \frac{6}{32} q_x k_y a^2 , & \frac{3}{32} q_x^2 a^2 + \frac{9}{32} q_y^2 a^2 \end{bmatrix} \vec{u}_0 = \frac{M\omega^2}{4K}\vec{u}_0 \quad \Big/ \cdot \frac{32}{3a^2}$$

$$\begin{bmatrix} 3q_x^2 + q_y^2 , & 2q_x q_y \\ 2q_x q_y , & q_x^2 + 3q_y^2 \end{bmatrix} \vec{u}_0 = \underbrace{\frac{8M\omega^2}{3Ka^2}}_{\lambda} \vec{u}_0$$

$$\begin{vmatrix} 3q_x^2 + q_y^2 - \lambda , & 2q_x q_y \\ 2q_x q_y , & q_x^2 + 3q_y^2 - \lambda \end{vmatrix} = 0$$

$$(3q_x^2 + q_y^2 - \lambda)(q_x^2 + 3q_y^2 - \lambda) - 4q_x^2 q_y^2 = 0$$

$$3q_x^4 + 9q_x^2 q_y^2 + q_y^2 q_x^2 + 3q_y^4 - \lambda(4q_x^2 + 4q_y^2) + \lambda^2 - 4q_x^2 q_y^2 = 0$$

$$\underbrace{3q_x^4 + 6q_x^2 q_y^2 + 3q_y^2}_{3(q_x^2 + q_y^2)^2} - 4\lambda(q_x^2 + q_y^2) + \lambda^2 = 0$$

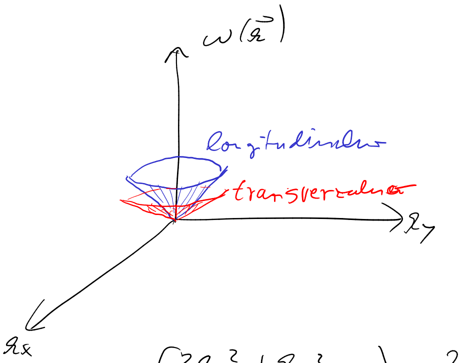
$$q^2 = q_x^2 + q_y^2 = |\vec{q}|^2$$

$$3q^4 - 4q^2\lambda + \lambda^2 = 0$$

$$(3q^2 - \lambda)(q^2 - \lambda) = 0 \longrightarrow \begin{matrix} \lambda_1 = 3q^2 \longrightarrow \omega^2 = \frac{3Ka^2}{8M} \cdot 3q^2 \longrightarrow \omega = \sqrt{\frac{9Ka^2}{8M}}\, q \\ \lambda_2 = q^2 \longrightarrow \omega^2 = \frac{3Ka^2}{8M} q^2 \longrightarrow \omega = \sqrt{\frac{3Ka^2}{8M}}\, q \end{matrix}$$

c_1, c_2

dve akustični veji



lastni vektorji:
$$\begin{bmatrix} 3q_x^2 + q_y^2 - \lambda , & 2q_x q_y \\ 2q_x q_y , & q_x^2 + 3q_y^2 - \lambda \end{bmatrix} \begin{bmatrix} u_{0x} \\ u_{0y} \end{bmatrix} = 0$$

$\lambda_1 = 3q^2$

$$\begin{bmatrix} 3q_x^2 + q_y^2 - (3q_x^2 + 3q_y^2) , & 2q_x q_y \end{bmatrix} \begin{bmatrix} u_{0x} \\ u_{0y} \end{bmatrix} = 0$$

$$\begin{bmatrix} -2q_y^2 , & 2q_x q_y \end{bmatrix} \begin{bmatrix} u_{0x} \\ u_{0y} \end{bmatrix} = 0 \quad \Big/ : 2k_y$$

$$\begin{bmatrix} -q_y , & q_x \end{bmatrix} \begin{bmatrix} u_{0x} \\ u_{0y} \end{bmatrix} = 0 \longrightarrow \begin{pmatrix} -q_y \\ q_x \end{pmatrix} \cdot \begin{pmatrix} q_x \\ q_y \end{pmatrix} = 0$$

$\vec{a} \cdot \vec{q} = 0$

$\vec{a} = \begin{pmatrix} -q_y \\ q_x \end{pmatrix}$

$\vec{a} \cdot \vec{u}_0 = 0$

$\left.\begin{matrix} \vec{a} \perp \vec{u}_0 \\ \vec{a} \perp \vec{q} \end{matrix}\right\} \vec{u}_0 \parallel \vec{q}$

longitudinalno valovanje

$\lambda_2 = q^2$

$$\begin{bmatrix} 3q_x^2 + \cancel{q_y^2} - q_x^2 - \cancel{q_y^2} , & 2q_x q_y \end{bmatrix} \begin{bmatrix} u_{0x} \\ u_{0y} \end{bmatrix} = 0$$

$$\begin{bmatrix} 2q_x^2 , & 2q_x q_y \end{bmatrix} \begin{bmatrix} u_{0x} \\ u_{0y} \end{bmatrix} = 0 \quad \Big/ : 2q_x$$

$$\begin{bmatrix} q_x , & q_y \end{bmatrix} \begin{bmatrix} u_{0x} \\ u_{0y} \end{bmatrix} = 0 \longrightarrow \vec{q} \cdot \vec{u}_0 = 0$$

transverzalno valovanje

$\vec{u}_0 \perp \vec{q}$