Naloga

Delec z vrtilno količino $l = 1$, ki se giblje v krogelno simetričnem potencialu, je v stanju $m = 1$. Ob $t = 0$ vklopimo homogeno magnetno polje B_0 pod kotom ϕ glede na os z. Kakšna je časovna odvisnost pričakovane vrednosti vrtilne količine?

Rešitev

Delec je ob času $t = 0$ v stanju $|lm\rangle = |11\rangle$. Smer spina je v z osi. Koordinatni sistem lahko obrnemo tako, da bo polje ležalo v XZ ravnini.

Zapišimo Hamiltonjan delca v polju

$$\begin{aligned} H &= -\boldsymbol{\mu} \cdot \mathbf{B} = \frac{\mu_b}{\hbar}\mathbf{L} \cdot \mathbf{B}, \quad \boldsymbol{\mu} = -\frac{\mu_b \mathbf{L}}{\hbar}, \quad \mu_b \text{ - Bohrov magneton} \\ H &= \frac{\mu_b}{\hbar}\left(L_x B_x + L_y B_y + L_z B_z\right) \end{aligned}$$

Sedaj pa naredimo transformacijo koordinatnega sistema v lastni koordinatni sistem polja.

$$\begin{aligned} (x, y, z) &\rightarrow (x', y', z') \\ (B_0 \sin\phi, 0, B_0 \cos\phi) &\rightarrow (0, 0, B_0) \end{aligned}$$

Na novo sedaj lahko zapišemo tudi hamiltonjana kot

$$H = \frac{\mu_b}{\hbar} B_0 L_z'$$

Razvijmo sedaj stanje $|11\rangle$ v novem sistemu $|lm\rangle'$, za katerega velja

$$\begin{aligned} L_z'|lm\rangle' &= \hbar m |lm\rangle', && (1) \\ L_x'|lm\rangle' &= \frac{1}{2}\left(L_+' + L_-'\right)|lm\rangle', && (2) \\ L_y'|lm\rangle' &= \frac{1}{2i}\left(L_+' - L_-'\right)|lm\rangle', && (3) \end{aligned}$$

pri čemer je $L_\pm|l, m\rangle = \sqrt{l(l+1) - m(m \pm 1)}\hbar|l, m \pm 1\rangle$.

L^2 - je skalar in tako neodvisen od izbire koordinat. $\rightarrow l' = 1$

Razvoj torej izgleda tako

$$|11\rangle = \sum_m A_m |1m\rangle' = A_0|10\rangle' + A_1|11\rangle' + A_{-1}|1-1\rangle'$$

$$\begin{aligned} L_z|11\rangle &= \hbar|11\rangle \\ L_z|11\rangle &= \left(L_z' \cos\phi - L_x' \sin\phi\right)\left(A_0|10\rangle' + A_1|11\rangle' + A_{-1}|1-1\rangle'\right) = \hbar\left(A_0|10\rangle' + A_1|11\rangle' + A_{-1}|1-1\rangle'\right) \end{aligned}$$

$$\left(-\hbar + L_z' \cos\phi - L_x' \sin\phi\right)\left(A_0|10\rangle' + A_1|11\rangle' + A_{-1}|1-1\rangle'\right) = 0 \qquad (4)$$

V tej enačbi moramo upoštevati še naslednji zvezi:

$$\begin{aligned} L_z'\left(A_0|10\rangle' + A_1|11\rangle' + A_{-1}|1-1\rangle'\right) &= \hbar\left(A_1|11\rangle' - A_{-1}|1-1\rangle'\right) \\ L_x'\left(A_0|10\rangle' + A_1|11\rangle' + A_{-1}|1-1\rangle'\right) &= \frac{A_0\hbar}{\sqrt{2}}\left(|11\rangle' + |1-1\rangle'\right) + \frac{(A_1 + A_{-1})\hbar}{\sqrt{2}}|10\rangle' \end{aligned}$$

ki ju izpeljemo s pomočjo enačb 1 in 2. Ko to vstavimo v enačbo 3, delujmo nanjo z operatorji $'\langle 10|$, $'\langle 11|$, $'\langle 1-1|$. Dobimo sistem treh enačb.

$$\begin{aligned} \langle' 10| : \quad & -A_0 + \tfrac{A_1}{\sqrt{2}}\sin\phi + \tfrac{A_{-1}}{\sqrt{2}}\sin\phi = 0 \\ \langle' 11| : \quad & -A_1 + A_1\cos\phi + \tfrac{A_0}{\sqrt{2}}\sin\phi = 0 \\ \langle' 1-1| : \quad & -A_{-1} - A_{-1}\cos\phi + \tfrac{A_0}{\sqrt{2}}\sin\phi = 0 \end{aligned}$$

Od tu dobimo zvezi

$$\begin{aligned} A_1 &= -\frac{A_0}{\sqrt{2}}\frac{\sin\phi}{1-\cos\phi} = -\frac{A_0}{\sqrt{2}}\cot\frac{\phi}{2} \\ A_{-1} &= -\frac{A_0}{\sqrt{2}}\tan\frac{\phi}{2} \end{aligned}$$

Faktor A_0 dobimo iz normalizacije. Veljati mora

$$\langle 11|11\rangle = \; |A_0|^2 \langle' 10|10\rangle' + |A_1|^2 \langle' 11|11\rangle' + |A_{-1}|^2 \langle' 1-1|1-1\rangle' = 1$$

$$\begin{aligned} 1 &= A_0^2 + A_{-1}^2 + A_1^2 = A_0^2\left(1 + \frac{1}{2}\cot^2\frac{\phi}{2} + \frac{1}{2}\tan^2\frac{\phi}{2}\right) \\ A_0^2 &= \left(1 + \frac{1}{2}\left(\frac{\cos^2\frac{\phi}{2}}{\sin^2\frac{\phi}{2}} + \frac{\sin^2\frac{\phi}{2}}{\cos^2\frac{\phi}{2}}\right)\right)^{-1} \\ A_0^2 &= \left(1 + \frac{1}{2}\left(\frac{\cos^4\frac{\phi}{2} + \sin^4\frac{\phi}{2}}{\sin^2\frac{\phi}{2}\cos^2\frac{\phi}{2}}\right)\right)^{-1} \\ A_0^2 &= 2 * \left(\frac{\cos^2\frac{\phi}{2} + \sin^2\frac{\phi}{2}}{\sin\frac{\phi}{2}\cos\frac{\phi}{2}}\right)^{-2} \\ A_0 &= \sqrt{2}\sin\frac{\phi}{2}\cos\frac{\phi}{2} \end{aligned}$$

To vstavimo nazaj v A_1 in A_{-1}

$$A_1 = -\cos^2\frac{\phi}{2}, \qquad A_{-1} = -\sin^2\frac{\phi}{2}$$

Naredimo sedaj časovni razvoj

$$E_1 = \frac{\mu_b B}{\hbar}(\hbar\cdot 1) = \mu_b B, \quad E_{-1} = -\mu_b B, \quad \omega = \frac{\mu_b B}{\hbar}$$

$$\begin{aligned} |\psi, t\rangle &= \sum e^{-i\frac{E_j}{\hbar}t}|j\rangle \\ &= -\cos^2\frac{\phi}{2}e^{-i\omega t}|11\rangle' - \sin^2\frac{\phi}{2}e^{i\omega t}|1-1\rangle' + \sqrt{2}\sin\frac{\phi}{2}\cos\frac{\phi}{2}|10\rangle' \end{aligned}$$

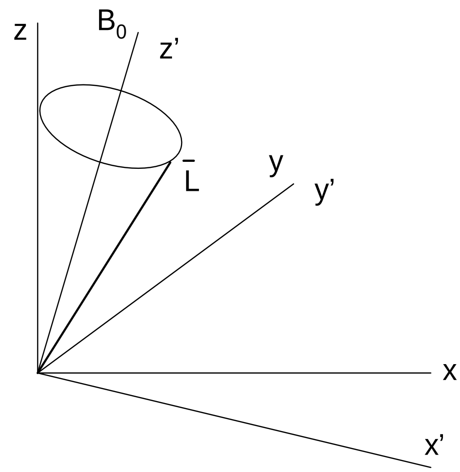
Sedaj ko vemo kakšna je valovna funkcija delca v času, si poglejmo kako se obnaša njegova vrtilna količina $\langle L(t)\rangle = ?$ Poglejmo posamezne komponente

$$\begin{aligned} \langle L_z'\rangle &= \langle\psi, t|L_z'|\psi, t\rangle \\ &= \cos^4\frac{\phi}{2}e^{i\omega t}\hbar e^{-i\omega t}\langle' 11|11\rangle' + \sin^4\frac{\phi}{2}e^{-i\omega t}(-\hbar)e^{i\omega t}\langle' 1-1|1-1\rangle' + 0 \\ &= \hbar\left(\cos^4\frac{\phi}{2} - \sin^4\frac{\phi}{2}\right) = \hbar\left(\cos^2\frac{\phi}{2} + \sin^2\frac{\phi}{2}\right)\left(\cos^2\frac{\phi}{2} - \sin^2\frac{\phi}{2}\right) \\ &= \hbar\cos\phi \end{aligned}$$

$$\langle L_x'\rangle = \mathrm{Re}\langle L_+'\rangle, \qquad \langle L_y'\rangle = \mathrm{Im}\langle L_+'\rangle$$

$$\begin{aligned} \langle L_+'\rangle &= 0 + \left(-\cos^2\frac{\phi}{2}\right)e^{i\omega t}2\hbar\sin\frac{\phi}{2}\cos\frac{\phi}{2}\langle' 11|11\rangle' + \left(-\sin^2\frac{\phi}{2}\right)e^{i\omega t}2\hbar\sin\frac{\phi}{2}\cos\frac{\phi}{2}\langle' 10|10\rangle' \\ &= -2\hbar\sin\frac{\phi}{2}\cos\frac{\phi}{2}e^{i\omega t} = -\hbar\sin\phi e^{i\omega t} \\ \langle L_x'\rangle &= -\hbar\sin\phi\cos(\omega t) \\ \langle L_y'\rangle &= -\hbar\sin\phi\sin(\omega t) \end{aligned}$$

Vrtilna količina torej precesira okoli smeri z'.


z
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