Domača naloga
Kvantna mehanika

DVONIVOJSKI SISTEM S ČASOVNO OMEJENO MOTNJO

Nuša Pukšič

Hamiltonjan prostega sistema:

$$\mathbf{H_0} = \begin{bmatrix} 0 & 0 \\ 0 & E_0 \end{bmatrix} \tag{1}$$

Lastni funkciji $\mathbf{H_0}$:

$$\mathbf{H_0}|\psi_1\rangle = 0 \tag{2}$$
$$\mathbf{H_0}|\psi_2\rangle = E_0|\psi_2\rangle \tag{3}$$

Hamiltonjan vzbujanega sistema:

$$\mathbf{H}' = \begin{bmatrix} 0 & V_0 \\ V_0 & E_0 \end{bmatrix} \tag{4}$$

Lastni vrednosti $\mathbf{H}'$:

$$\det \mathbf{H}' = -\lambda(E_0 - \lambda) - V_0^2 = 0 \tag{5}$$
$$\lambda^2 - \lambda E_0 - V_0^2 = 0 \tag{6}$$

$$\lambda_{1,2} = \frac{E_0}{2}\left(1 \pm \sqrt{1 + 4\Big(\frac{V_0}{E_0}\Big)^2}\,\right) \tag{7}$$

Lastni funkciji $\mathbf{H}'$:

$$\mathbf{H}'|\psi_1'\rangle = \lambda_1|\psi_1'\rangle \tag{8}$$
$$\mathbf{H}'|\psi_2'\rangle = \lambda_2|\psi_2'\rangle \tag{9}$$

$|\psi(0)\rangle$ lahko razvijemo po lastnih funkcijah $\mathbf{H}$ (vemo, da sistem v osnovnem stanju) ali $\mathbf{H}'$:

$$|\psi(0)\rangle = |\psi_1\rangle \tag{10}$$
$$|\psi(0)\rangle = \alpha_1|\psi_1'\rangle + \alpha_2|\psi_2'\rangle \tag{11}$$

Izrazimo najprej eno bazo z drugo:

$$|\psi_1'\rangle = c_1|\psi_1\rangle + c_2|\psi_2\rangle \tag{12}$$
$$|\psi_2'\rangle = d_1|\psi_1\rangle + d_2|\psi_2\rangle \tag{13}$$

In nato še obratno:

$$|\psi_1\rangle = \frac{d_2}{d_2c_1 - d_1c_2}|\psi_1'\rangle - \frac{c_2}{d_2c_1 - d_1c_2}|\psi_2'\rangle \tag{14}$$

$$|\psi_2\rangle = -\frac{d_1}{d_2c_1 - d_1c_2}|\psi_1'\rangle + \frac{c_1}{d_2c_1 - d_1c_2}|\psi_2'\rangle \tag{15}$$

Matriko prehoda med bazama označimo z $\mathcal{O}$:

$$\mathcal{O} = \begin{bmatrix} c_1 & c_2 \\ d_1 & d_2 \end{bmatrix} \tag{16}$$

Ker je $\mathcal{O}$ unitarna, velja $\mathcal{O}^{-1} = \mathcal{O}^T$, iz česar sledi:

$$c_1 = d_2 \tag{17}$$

$$d_1 = -c_2 \tag{18}$$

$$d_2c_1 - d_1c_2 = c_1^2 + c_2^2 = 1 \tag{19}$$

Primerjamo enačbe (10), (11) in (14) in zaključimo:

$$\alpha_1 = \frac{d_2}{d_2c_1 - d_1c_2} = c_1 \tag{20}$$

$$\alpha_2 = -\frac{c_2}{d_2c_1 - d_1c_2} = -c_2 \tag{21}$$

Naredimo časovni razvoj $|\psi\rangle$ po lastnih funkcijah $\mathbf{H}'$:

$$|\psi(t_0)\rangle = c_1|\psi_1'(t_0)\rangle - c_2|\psi_2'(t_0)\rangle = \tag{22}$$

$$= c_1|\psi_1'\rangle \exp(-i\frac{\lambda_1}{\hbar}t_0) - c_2|\psi_2'\rangle \exp(-i\frac{\lambda_2}{\hbar}t_0) \tag{23}$$

Spet zamenjamo bazo:

$$|\psi(t_0)\rangle = c_1(c_1|\psi_1\rangle + c_2|\psi_2\rangle) \exp(-i\tfrac{\lambda_1}{\hbar}t_0) \quad - \tag{24}$$

$$- \quad c_2(d_1|\psi_1\rangle + d_2|\psi_2\rangle) \exp(-i\tfrac{\lambda_2}{\hbar}t_0) \tag{25}$$

$$|\psi(t_0)\rangle = (c_1^2 \exp(-i\tfrac{\lambda_1}{\hbar}t_0) + c_2^2 \exp(-i\tfrac{\lambda_2}{\hbar}t_0))|\psi_1\rangle \quad + \tag{26}$$

$$+ \quad (c_1c_2 \exp(-i\tfrac{\lambda_1}{\hbar}t_0) - c_1c_2 \exp(-i\tfrac{\lambda_2}{\hbar}t_0))|\psi_2\rangle \tag{27}$$

Vpeljemo novo oznako:

$$A = c_1c_2 \tag{28}$$

Izračunamo verjetnost, da sistem po času t_0 v vzbujenem stanju:

$$P = \left| A\Big(\exp(-i\frac{\lambda_1}{\hbar}t_0) - \exp(-i\frac{\lambda_2}{\hbar}t_0)\Big)\right|^2 = \quad (29)$$

$$= 2A^2\Big(1 - cos(\frac{\lambda_2 - \lambda_1}{\hbar}t_0)\Big) \quad (30)$$

Verjetnost je največja, ko je:

$$cos(\frac{\lambda_2 - \lambda_1}{\hbar}t_0) = -1 \quad (31)$$

Iz tega sledi:

$$t_0 = \frac{\pi\hbar}{\lambda_2 - \lambda_1} \quad (32)$$

$$P_{max} = 4A^2 \quad (33)$$

Kdaj je verjetnost lahko enaka 1?

$$P = 4A^2 = 1 \quad (34)$$

$$A = c_1 d_1 = \pm\frac{1}{2} \quad (35)$$

$$c_1^2 + d_1^2 = 1 \Rightarrow c_1 = \pm d_1 = \frac{1}{\sqrt{2}} \quad (36)$$

$$\begin{bmatrix} 0 & V_0 \\ V_0 & E_0 \end{bmatrix} \begin{bmatrix} c_1 \\ d_1 \end{bmatrix} = \lambda \begin{bmatrix} \frac{1}{\sqrt{2}} \\ \pm\frac{1}{\sqrt{2}} \end{bmatrix} \quad (37)$$

Če je $V_0 \gg E_0$ lahko zapišemo:

$$\lambda_{1,2} = \frac{E_0}{2}\left(1 \pm \sqrt{1 + 4\Big(\frac{V_0}{E_0}\Big)^2}\right) = \quad (38)$$

$$= \frac{E_0}{2}\Big(1 \pm 2\frac{V_0}{E_0}\Big) \Rightarrow \pm V_0 \quad (39)$$

$$(40)$$

Tedaj gre $t_0 \Rightarrow \pi\hbar/2V_0$.