Vaje iz kvantne mehanike 1

Nedoločenost koherentnega stanja

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Jela Škerlak

Naloga

Za koherentno stanje harmonskega oscilatorja poišči nedoločenost koordinate, gibalne količine, potencialne energije, kinetične energije in celotne energije.

Rešitev

Iskane nedoločenosti so

(a) $\Delta x = \sqrt{\langle x^2 \rangle - \langle x \rangle^2}$

(b) $\Delta p = \sqrt{\langle p^2 \rangle - \langle p \rangle^2}$

(c) $\Delta E_{pot} = \sqrt{\langle \frac{k^2 x^4}{4} \rangle - \langle \frac{kx^2}{2} \rangle^2}$

(d) $\Delta E_{kin} = \sqrt{\langle \frac{p^4}{4m^2} \rangle - \langle \frac{p^2}{2m} \rangle^2}$

(e) $\Delta E = \sqrt{\langle \hbar^2 \omega^2 (a^\dagger a + \frac{1}{2})^2 \rangle - \langle \hbar\omega (a^\dagger a + \frac{1}{2}) \rangle^2}$

Torej je treba izračunati $\langle x \rangle, \langle x^2 \rangle, \langle p \rangle, \langle p^2 \rangle, \langle x^4 \rangle, \langle p^4 \rangle, \langle E \rangle$ in $\langle E^2 \rangle$ za koherentno stanje

$$|\alpha\rangle = e^{-\frac{1}{2}|\alpha|^2} \sum_n \frac{\alpha^n}{n!} |n\rangle,$$

kjer velja

$$\begin{aligned} x &= \sqrt{\frac{\hbar}{2m\omega}}(a + a^\dagger) \\ p &= -i\sqrt{\frac{m\omega\hbar}{2}}(a - a^\dagger) \\ E &= \hbar\omega(a^\dagger a + \frac{1}{2}) \\ a|\alpha\rangle &= \alpha|\alpha\rangle \\ \langle\alpha|a^\dagger &= \langle\alpha|\alpha^* \\ [a, a^\dagger] &= 1. \end{aligned}$$

(a) $\langle x\rangle = \langle\alpha|\sqrt{\frac{\hbar}{2m\omega}}(a+a^\dagger)|\alpha\rangle = \sqrt{\frac{\hbar}{2m\omega}}(\alpha+\alpha^*)$, in tako dobimo

$$\langle x\rangle^2 = \frac{\hbar}{2m\omega}(\alpha+\alpha^*)^2.$$

Za $\langle x^2\rangle$ najprej izračunamo operator x^2 s pomočjo $[a,a^\dagger]=1$:

$$x^2 = \frac{\hbar}{2m\omega}(a^2+(a^\dagger)^2+\underbrace{aa^\dagger}_{=1+a^\dagger a}+a^\dagger a) = \frac{\hbar}{2m\omega}(a^2+(a^\dagger)^2+2a^\dagger a+1)$$

Torej

$$\begin{aligned}\langle x^2\rangle &= \langle\alpha|\frac{\hbar}{2m\omega}(a^2+(a^\dagger)^2+2a^\dagger a+1)|\alpha\rangle\\ &= \frac{\hbar}{2m\omega}(\alpha^2+(\alpha^*)^2+2\alpha^*\alpha+1)\\ &= \frac{\hbar}{2m\omega}((\alpha+\alpha^*)^2+1)\end{aligned}$$

Za nedoločenost potem sledi

$$\Delta x = \sqrt{\langle x^2\rangle - \langle x\rangle^2} = \sqrt{\frac{\hbar}{2m\omega}((\alpha+\alpha^*)^2+1) - \frac{\hbar}{2m\omega}(\alpha+\alpha^*)^2} = \sqrt{\frac{\hbar}{2m\omega}}.$$

(b) Z istim postopkom kot za (a) dobimo

$$\langle p\rangle^2 = -\frac{m\omega\hbar}{2}(\alpha-\alpha^*)^2,$$

$$p^2 = -\frac{m\omega\hbar}{2}(a^2+(a^\dagger)^2-\underbrace{aa^\dagger}_{=1+a^\dagger a}-a^\dagger a) = -\frac{m\omega\hbar}{2}(a^2+(a^\dagger)^2-1-2a^\dagger a)$$

in torej

$$\langle p^2\rangle = -\frac{m\omega\hbar}{2}(\alpha^2+(\alpha^*)^2-1-2\alpha^*\alpha) = -\frac{m\omega\hbar}{2}((\alpha-(\alpha^*))^2-1).$$

Nedoločenost gibalne količine je potem

$$\Delta p = \sqrt{\langle p^2\rangle - \langle p\rangle^2} = \sqrt{-\frac{m\omega\hbar}{2}((\alpha-(\alpha^*))^2-1) + \frac{m\omega\hbar}{2}(\alpha-\alpha^*)^2} = \sqrt{\frac{m\omega\hbar}{2}}.$$

Takoj lahko preverimo, da velja

$$\Delta x\Delta p = \sqrt{\frac{\hbar}{2m\omega}}\sqrt{\frac{m\omega\hbar}{2}} = \frac{\hbar}{2}.$$

(c) Za potencialno energijo je postopek isti, ampak treba je več računati. Namreč moramo določati x^4 kot operator z a in $a^\dagger$ tako, da imamo vedno $a^\dagger$ na levi in a na desni strani ko izračunamo $\langle\alpha|x^4|\alpha\rangle$. Zaradi prostora pustimo konstanto pred x.

$$\begin{aligned} x^4 &= x^2x^2 = (a^2 + (a^\dagger)^2 + 2a^\dagger a + 1)^2 \\ &= a^4 + (a^\dagger)^4 + 2(a^\dagger)^3 a + 2a^\dagger a^3 + 2(a^\dagger)^2 + 2a^2 + (a^\dagger)^2a^2 + 4a^\dagger a + 1 \\ &+ \underbrace{a^2(a^\dagger)^2}_{:=b} + \underbrace{2a^2a^\dagger a}_{:=c} + \underbrace{2a^\dagger a(a^\dagger)^2}_{:=d} + \underbrace{4a^\dagger aa^\dagger a}_{:=e} \end{aligned}$$

Člene b, c, d in e je torej treba spremeniti.

$$\begin{aligned} b &= a^2(a^\dagger)^2 = a(aa^\dagger)a^\dagger = a(1 + a^\dagger a)a^\dagger = aa^\dagger + aa^\dagger aa^\dagger \\ &= 1 + a^\dagger a + (1 + a^\dagger a)(1 + a^\dagger a) \\ &= 1 + a^\dagger a + 1 + 2a^\dagger a + a^\dagger aa^\dagger a \\ &= 3a^\dagger a + 2 + a^\dagger(1 + a^\dagger a)a \\ &= 3a^\dagger a + 2 + a^\dagger a + (a^\dagger)^2a^2 = 4a^\dagger a + 2 + (a^\dagger)^2a^2 \end{aligned}$$

Na podoben način dobimo

$$\begin{aligned} c &= 2a^2a^\dagger a = 4a^2 + 2a^\dagger a^3 \\ d &= 2a^\dagger a(a^\dagger)^2 = 4(a^\dagger)^2 + 2(a^\dagger)^3 a \\ e &= 4a^\dagger aa^\dagger a = 4a^\dagger a + 4(a^\dagger)^2a^2. \end{aligned}$$

Končno

$$x^4 = a^4 + (a^\dagger)^4 + 4(a^\dagger)^3a + 4a^\dagger a^3 + 6(a^\dagger)^2 + 6a^2 + 6(a^\dagger)^2a^2 + 12a^\dagger a + 3.$$

Od tod sledi (sedaj spet s konstanto)

$$\langle x^4\rangle = \left(\frac{\hbar}{2m\omega}\right)^2 (\alpha^4 + (\alpha^*)^4 + 4(\alpha^*)^3\alpha + 4\alpha^*\alpha^3 + 6(\alpha^*)^2 + 6\alpha^2 + 6(\alpha^*)^2\alpha^2 + 12\alpha^*\alpha + 3).$$

Izračunajmo še $\langle x^2\rangle^2$:

$$\begin{aligned} \langle x^2\rangle^2 &= \left(\frac{\hbar}{2m\omega}\right)^2 ((\alpha + \alpha^*)^2 + 1)^2 = \left(\frac{\hbar}{2m\omega}\right)^2 ((\alpha + \alpha^*)^4 + 2(\alpha + \alpha^*)^2 + 1) \\ &= \left(\frac{\hbar}{2m\omega}\right)^2 (\alpha^4 + (\alpha^*)^4 + 4\alpha^*\alpha^3 + 4(\alpha^*)^3\alpha + 6(\alpha^*)^2\alpha^2 + 4\alpha^*\alpha + 2\alpha^2 + 2(\alpha^*)^2 + 1) \end{aligned}$$

Torej

$$\begin{aligned} \Delta E_{pot} &= \sqrt{\langle\frac{k^2x^4}{4}\rangle - \langle\frac{kx^2}{2}\rangle^2} \\ &= \left(\frac{\hbar}{2m\omega}\right)\frac{k}{2}\sqrt{(4\alpha^2 + 4(\alpha^*)^2 + 8\alpha^*\alpha + 2)} \\ &\overset{k=m\omega^2}{=} \left(\frac{\hbar\omega}{2}\right)\sqrt{((\alpha + \alpha^*)^2 + \frac{1}{2})}. \end{aligned}$$

(d) Na isti način kot za potencialno energijo dobimo nedoločenost kinetične energije:

$$\langle p^4\rangle = \left(\frac{m\omega\hbar}{2}\right)^2 (\alpha^4+(\alpha^*)^4-4(\alpha^*)^3\alpha-4\alpha^*\alpha^3-6(\alpha^*)^2-6\alpha^2+6(\alpha^*)^2\alpha^2+12\alpha^*\alpha+3)$$

$$\begin{aligned}\langle p^2\rangle^2 &= \left(\frac{m\omega\hbar}{2}\right)^2 ((\alpha-\alpha^*)^2-1)^2 \\ &= \left(\frac{m\omega\hbar}{2}\right)^2 (\alpha^4+(\alpha^*)^4-4(\alpha^*)^3\alpha-4\alpha^*\alpha^3+6(\alpha^*)^2\alpha^2-2\alpha^2-2(\alpha^*)^2+4\alpha^*\alpha+1)\end{aligned}$$

Torej

$$\begin{aligned}\Delta E_{kin} &= \left(\frac{m\omega\hbar}{4m}\right)\sqrt{\langle p^4\rangle-\langle p^2\rangle^2} \\ &= \left(\frac{\omega\hbar}{4}\right)\sqrt{-4((\alpha-\alpha^*)^2-\frac{1}{2})} \\ &= \left(\frac{\omega\hbar}{2}\right)\sqrt{\frac{1}{2}-(\alpha-\alpha^*)^2}\end{aligned}$$

(e) $E=\hbar\omega(a^\dagger a+\frac{1}{2})$, torej

$$\langle E\rangle^2 = \hbar^2\omega^2(\langle\alpha|a^\dagger a+\frac{1}{2}|\alpha\rangle)^2 = \hbar^2\omega^2(\alpha^*\alpha+\frac{1}{2})^2.$$

Treba je še izračunati E^2:

$$\begin{aligned}E^2 &= \hbar^2\omega^2(a^\dagger a+\frac{1}{2})^2 \\ &= \hbar^2\omega^2(a^\dagger a a^\dagger a+a^\dagger a+\frac{1}{4}) \\ &= \hbar^2\omega^2(a^\dagger(1+a^\dagger a)a+a^\dagger a+\frac{1}{4}) \\ &= \hbar^2\omega^2(2a^\dagger a+(a^\dagger)^2a^2+\frac{1}{4})\end{aligned}$$

Od tod sledi

$$\langle E^2\rangle = \hbar^2\omega^2(2\alpha^*\alpha+(\alpha^*)^2\alpha^2+\frac{1}{4}) = \hbar^2\omega^2((\alpha^*\alpha+\frac{1}{2})^2+\alpha^*\alpha)$$

in končno

$$\Delta E = \hbar\omega\sqrt{\langle E^2\rangle-\langle E\rangle^2} = \hbar\omega\sqrt{\alpha^*\alpha}.$$