Operator rotacije

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1 Naloga

Zapiši matriko operatorja rotacije okrog x osi za kot φ v bazi lastnih stanj operatorja $\hat{L}^2$ z lastno vrednostjo $\hbar^2 l(l+1), l = 1$ in opearatorja L_z. S to rešitvijo eksplicitno pokaži, da se v sistemu s Hamiltonovim operatorjem $\hat{H} = aL_x$ stanje, ki ima ob času 0 dobro določeno komponento vrtilne količine v z smeri, po določenem času obrne v stanje z nasprotno komponento vrtilne količine.

2 Rešitev

Smo v bazi $l = 1, m = -1, 0, 1$
Operator rotacije zapišemo kot:

$$U_{\vec{\varphi}} = e^{-i\frac{\vec{\varphi}\vec{L}}{\hbar}}$$

Vrtimo okoli osi x, zato zapišemo:

$$\vec{\varphi} = (\varphi, 0, 0)$$

$$\vec{L} = (L_x, L_y, L_z)$$

Naš operator lahko zapišemo z vsoto:

$$U_{\vec{\varphi}} = e^{-i\frac{\varphi L_x}{\hbar}} = \sum_{n=0}^{\infty} \frac{1}{n!} \left(-i\frac{\varphi L_x}{\hbar}\right)^n$$

Zračunajmo L_x:

$$L_{\pm} = L_x \pm iL_y$$

$$L_+^+ = L_-$$

$$L_+|l, m\rangle = \hbar\sqrt{l(l+1) - m(m+1)}|l, m+1\rangle$$

$$L_-|l, m\rangle = \hbar\sqrt{l(l+1) - m(m-1)}|l, m-1\rangle$$

Edina neničelna matrična elementa sta.

$$\langle 0|L_+|-1\rangle = \langle 0|\hbar\sqrt{2}|0\rangle = \hbar\sqrt{2}$$

$$\langle 1|L_+|0\rangle = \langle 1|\hbar\sqrt{2}|1\rangle = \hbar\sqrt{2}$$

$$L_+ = \begin{bmatrix} 0 & 0 & 0 \\ \hbar\sqrt{2} & 0 & 0 \\ 0 & \hbar\sqrt{2} & 0 \end{bmatrix} \qquad L_- = L_+^T = \begin{bmatrix} 0 & \hbar\sqrt{2} & 0 \\ 0 & 0 & \hbar\sqrt{2} \\ 0 & 0 & 0 \end{bmatrix}$$

$$L_\pm = L_x \pm iL_y \qquad L_x = \frac{L_+ + L_-}{2}$$

$$L_\pm = L_x \pm iL_y \qquad L_y = \frac{L_+ - L_-}{2i}$$

Torej seštejemo matriki L_+ in L_- in uvedemo še matriko M:

$$L_x = \begin{bmatrix} 0 & \frac{\hbar\sqrt{2}}{2} & 0 \\ \frac{\hbar\sqrt{2}}{2} & 0 & \frac{\hbar\sqrt{2}}{2} \\ 0 & \frac{\hbar\sqrt{2}}{2} & 0 \end{bmatrix} \qquad L_x = \hbar M$$

Če pogledamo naš operator rotacije vidimo da bomo potrebovali potence matrike M:

$$U_{\vec{\varphi}} = \sum_{n=0}^{\infty} \frac{1}{n!}\left(-i\frac{\varphi L_x}{\hbar}\right)^n = \sum_{n=0}^{\infty} \frac{1}{n!}\left(-i\varphi M\right)^n$$

Opazimo tole:

$$M^2 = \begin{bmatrix} 0 & \frac{\sqrt{2}}{2} & 0 \\ \frac{\sqrt{2}}{2} & 0 & \frac{\sqrt{2}}{2} \\ 0 & \frac{\sqrt{2}}{2} & 0 \end{bmatrix}^2 = \begin{bmatrix} 0 & \frac{1}{2} & 0 \\ \frac{1}{2} & 0 & \frac{1}{2} \\ 0 & \frac{1}{2} & 0 \end{bmatrix} = M^{2n}$$

$$M^3 = M^2 M = \begin{bmatrix} 0 & \frac{\sqrt{2}}{2} & 0 \\ \frac{\sqrt{2}}{2} & 0 & \frac{\sqrt{2}}{2} \\ 0 & \frac{\sqrt{2}}{2} & 0 \end{bmatrix}^2 = M = M^{(2n+1)}$$

Vsoto, s katero smo prej zapisali operator rotacije lahko tako razdelimo na sode in lihe člene:

$$U_{\vec{\varphi}} = \sum_{n=0}^{\infty} \frac{1}{n!} (-i\varphi M)^n = I + M \sum_{n=lih}^{\infty} \frac{1}{n!} (-i\varphi)^n + M^2 \sum_{n=sod, n\neq 0}^{\infty} \frac{1}{n!} (-i\varphi)^n$$

Za sode n uvedemo $n = 2k$ in za lihe n $n = 2k + 1$ in dobimo dve znani vsoti:

$$\begin{aligned} U_{\vec{\varphi}} &= I + M \sum_{k=0}^{\infty} \frac{1}{(2k+1)!} (-i\varphi)^{2k+1} + M^2 \sum_{k=1}^{\infty} \frac{1}{2k!} (-i\varphi)^{2k} \\ &= I + M(-i) \underbrace{\sum_{k=0}^{\infty} \frac{1}{(2k+1)!} (-1)^k (\varphi)^{2k+1}}_{\sin\varphi} + M^2 \underbrace{\sum_{k=1}^{\infty} \frac{1}{2k!} (-1)^k (\varphi)^{2k}}_{\cos\varphi - 1} \\ &= I - iM \sin\varphi + M^2(\cos\varphi - 1) \end{aligned}$$

$$U_{\vec{\varphi}} = \begin{bmatrix} \frac{1}{2}\cos\varphi + \frac{1}{2} & -i\frac{\sqrt{2}}{2}\sin\varphi & \frac{1}{2}(\cos\varphi - 1) \\ -i\frac{\sqrt{2}}{2}\sin\varphi & \cos\varphi & -i\frac{\sqrt{2}}{2}\sin\varphi \\ \frac{1}{2}(\cos\varphi - 1) & -i\frac{\sqrt{2}}{2}\sin\varphi & \frac{1}{2}\cos\varphi + \frac{1}{2} \end{bmatrix}$$

Stanje v sistemu s hamiltonovim operatorjem $\hat{H} = aL_x$ ima ob času 0 dobro določeno komponento vrtilne količine v z smeri. Kdaj se stanje obrne v nasprotno smer?

$$|\psi_{(t=0)}\rangle = |1, 1\rangle \qquad \hat{H} = aL_x$$

$$U_t = e^{-\frac{i\hat{H}t}{\hbar}} = e^{-\frac{iaL_x t}{\hbar}} = e^{-i\frac{\vec{\varphi}\vec{L}}{\hbar}}$$

Vidimo da je $\varphi = at$

$$|1, 1\rangle = \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}$$

$$U_t|1, 1\rangle = \begin{bmatrix} \frac{1}{2}(\cos\varphi - 1) \\ -i\frac{\sqrt{2}}{2}\sin\varphi \\ \frac{1}{2}(\cos\varphi + 1) \end{bmatrix} = \begin{bmatrix} \frac{1}{2}(-1 - 1) \\ 0 \\ 0 \end{bmatrix} = \begin{bmatrix} -1 \\ 0 \\ 0 \end{bmatrix}$$

Vidimo da je to res če je:

$$\varphi = \pi + 2k\pi$$

$$\varphi = at \qquad t = \frac{\varphi}{a} = \frac{\pi}{a}$$