Domača naloga pri predmetu Kvantna mehanika I

Heisenbergova interakcija za $S = 3/2$

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Naloga

Poišči lastne energije za operator Heisenbergove interakcije $H = -J\mathbf{S_1}\cdot\mathbf{S_2}$, kjer ima vsak od spinov kvantno število velikosti vrtilne količine $S = 3/2$. Za (nekatera) lastna stanja sistema izračunaj pričakovane vrednosti in nedoločenosti komponent vrtilne količine v z smeri za posamezna spina.

Rešitev

V operatorju $H = -J\mathbf{S_1}\cdot\mathbf{S_2}$ nastopa skalarni produkt obeh spinov. Skupni spin je $\mathbf{S} = \mathbf{S_1} + \mathbf{S_2}$, njegov kvadrat pa se glasi $\mathbf{S}^2 = \mathbf{S_1}^2 + \mathbf{S_2}^2 + 2\,\mathbf{S_1}\cdot\mathbf{S_2}$. Od tod lahko izrazimo $\mathbf{S_1}\cdot\mathbf{S_2} = \frac{1}{2}(\mathbf{S}^2 - \mathbf{S_1}^2 - \mathbf{S_2}^2)$ iz česar dobimo operator energije v obliki

$$H = -\frac{J}{2}(\mathbf{S}^2 - \mathbf{S_1}^2 - \mathbf{S_2}^2). \tag{1}$$

Lastna stanja za ta operator opišemo z velikostjo skupne vrtilne količine S in njeno projekcijo na os z, to je m. Vsako tako stanje lahko zapišemo kot vsoto različnih stanj obeh delcev, za katera mora veljati $|S_1 - S_2| \leq S \leq S_1 + S_2$ in $m = m_1 + m_2$, kjer sta S_1 ter S_2 velikosti obeh spinov in m_1 ter m_2 velikosti njunih projekcij na z os. Utežne koeficiente za vsako tako linearno kombinacijo stanj dobimo v Clebsch-Gordanovi tabeli koeficientov. S pomočjo razdelka $3/2 \times 3/2$ zapišemo vse možne lastne funkcije:

1. $|S = 3, m = 3\rangle = |m_1 = \frac{3}{2}, m_2 = \frac{3}{2}\rangle$
2. $|S = 3, m = 2\rangle = \sqrt{\frac{1}{2}}|m_1 = \frac{3}{2}, m_2 = \frac{1}{2}\rangle + \sqrt{\frac{1}{2}}|m_1 = \frac{1}{2}, m_2 = \frac{3}{2}\rangle$
3. $|S = 2, m = 2\rangle = \sqrt{\frac{1}{2}}|m_1 = \frac{3}{2}, m_2 = \frac{1}{2}\rangle - \sqrt{\frac{1}{2}}|m_1 = \frac{1}{2}, m_2 = \frac{3}{2}\rangle$
4. $|S = 3, m = 1\rangle = \sqrt{\frac{1}{5}}|m_1 = \frac{3}{2}, m_2 = -\frac{1}{2}\rangle + \sqrt{\frac{3}{5}}|m_1 = \frac{1}{2}, m_2 = \frac{1}{2}\rangle + \sqrt{\frac{1}{5}}|m_1 = -\frac{1}{2}, m_2 = \frac{3}{2}\rangle$
5. $|S = 2, m = 1\rangle = \sqrt{\frac{1}{2}}|m_1 = \frac{3}{2}, m_2 = -\frac{1}{2}\rangle - \sqrt{\frac{1}{5}}|m_1 = -\frac{1}{2}, m_2 = \frac{3}{2}\rangle$

6. $|S=1, m=1\rangle = \sqrt{\frac{3}{10}}|m_1=\frac{3}{2}, m_2=-\frac{1}{2}\rangle - \sqrt{\frac{2}{5}}|m_1=\frac{1}{2}, m_2=\frac{1}{2}\rangle + \sqrt{\frac{3}{10}}|m_1=-\frac{1}{2}, m_2=\frac{3}{2}\rangle$

7. $|S=3, m=0\rangle = \sqrt{\frac{1}{20}}|m_1=\frac{3}{2}, m_2=-\frac{3}{2}\rangle + \sqrt{\frac{9}{20}}|m_1=\frac{1}{2}, m_2=-\frac{1}{2}\rangle + \sqrt{\frac{9}{20}}|m_1=-\frac{1}{2}, m_2=\frac{1}{2}\rangle + \sqrt{\frac{1}{20}}|m_1=-\frac{3}{2}, m_2=\frac{3}{2}\rangle$

8. $|S=2, m=0\rangle = \sqrt{\frac{1}{4}}|m_1=\frac{3}{2}, m_2=-\frac{3}{2}\rangle + \sqrt{\frac{1}{4}}|m_1=\frac{1}{2}, m_2=-\frac{1}{2}\rangle - \sqrt{\frac{1}{4}}|m_1=-\frac{1}{2}, m_2=\frac{1}{2}\rangle - \sqrt{\frac{1}{4}}|m_1=-\frac{3}{2}, m_2=\frac{3}{2}\rangle$

9. $|S=1, m=0\rangle = \sqrt{\frac{9}{20}}|m_1=\frac{3}{2}, m_2=-\frac{3}{2}\rangle - \sqrt{\frac{1}{20}}|m_1=\frac{1}{2}, m_2=-\frac{1}{2}\rangle - \sqrt{\frac{1}{20}}|m_1=-\frac{1}{2}, m_2=\frac{1}{2}\rangle + \sqrt{\frac{9}{20}}|m_1=-\frac{3}{2}, m_2=\frac{3}{2}\rangle$

10. $|S=0, m=0\rangle = \sqrt{\frac{1}{4}}|m_1=\frac{3}{2}, m_2=-\frac{3}{2}\rangle - \sqrt{\frac{1}{4}}|m_1=\frac{1}{2}, m_2=-\frac{1}{2}\rangle + \sqrt{\frac{1}{4}}|m_1=-\frac{1}{2}, m_2=\frac{1}{2}\rangle - \sqrt{\frac{1}{4}}|m_1=-\frac{3}{2}, m_2=\frac{3}{2}\rangle$

11. $|S=3, m=-1\rangle = \sqrt{\frac{1}{5}}|m_1=\frac{1}{2}, m_2=-\frac{3}{2}\rangle + \sqrt{\frac{3}{5}}|m_1=-\frac{1}{2}, m_2=-\frac{1}{2}\rangle + \sqrt{\frac{1}{5}}|m_1=-\frac{3}{2}, m_2=\frac{1}{2}\rangle$

12. $|S=2, m=-1\rangle = \sqrt{\frac{1}{2}}|m_1=\frac{1}{2}, m_2=-\frac{3}{2}\rangle - \sqrt{\frac{1}{2}}|m_1=-\frac{3}{2}, m_2=\frac{1}{2}\rangle$

13. $|S=1, m=-1\rangle = \sqrt{\frac{3}{10}}|m_1=\frac{1}{2}, m_2=-\frac{3}{2}\rangle - \sqrt{\frac{2}{5}}|m_1=-\frac{1}{2}, m_2=-\frac{1}{2}\rangle + \sqrt{\frac{3}{10}}|m_1=-\frac{3}{2}, m_2=\frac{1}{2}\rangle$

14. $|S=3, m=-2\rangle = \sqrt{\frac{1}{2}}|m_1=-\frac{1}{2}, m_2=-\frac{3}{2}\rangle + \sqrt{\frac{1}{2}}|m_1=-\frac{3}{2}, m_2=-\frac{1}{2}\rangle$

15. $|S=2, m=-2\rangle = \sqrt{\frac{1}{2}}|m_1=-\frac{1}{2}, m_2=-\frac{3}{2}\rangle - \sqrt{\frac{1}{2}}|m_1=-\frac{3}{2}, m_2=-\frac{1}{2}\rangle$

16. $|S=3, m=-3\rangle = |m_1=-\frac{3}{2}, m_2=-\frac{3}{2}\rangle$

Iz enačbe (1) dobimo, če upoštevamo $S_1 = S_2 = \frac{3}{2}$, za lastne energije sledeč izraz:

$$H|S,m\rangle = -\frac{J\hbar^2}{2}\Big(S\Big(S+1\Big) - \frac{3}{2}\Big(\frac{3}{2}+1\Big) - \frac{3}{2}\Big(\frac{3}{2}+1\Big)\Big). \tag{2}$$

Iz enačbe je razvidno, da je energija posameznega lastnega stanja odvisna le od velikosti skupnega spina S. Za zapisana stanja dobimo lastne energije:

Energija	Lastna stanja (zaporedne številke)
$\frac{15}{2}\frac{J\hbar^2}{2}$	10.
$\frac{11}{2}\frac{J\hbar^2}{2}$	6. , 9. , 13.
$\frac{3}{2}\frac{J\hbar^2}{2}$	3. , 5. , 8. , 12. , 15.
$-\frac{9}{2}\frac{J\hbar^2}{2}$	1. , 2. , 4. , 7. , 11. , 14. , 16.

Na koncu izračunamo še nekaj pričakovanih vrednosti in nedoločenosti komponent vrtilne količine v z smeri za posamezna spina. Pričakovane vrednosti so

$$\langle \text{stanje}|S_{i\,z}|\text{stanje}\rangle, \quad \langle \text{stanje}|S_{i\,z}^2|\text{stanje}\rangle, \quad i = 1, 2,$$

nedoločenosti pa

$$\delta S_{i\,z}^2 = \langle \text{stanje}|S_{i\,z}^2|\text{stanje}\rangle - \langle \text{stanje}|S_{i\,z}|\text{stanje}\rangle^2.$$

Za stanje $S = 3$, $m = 3$ za prvi spin dobimo:

$$\langle S = 3, m = 3|S_{1\,z}|S = 3, m = 3\rangle = \left\langle m_1 = \frac{3}{2}, m_2 = \frac{3}{2}\middle|S_{1\,z}\middle|m_1 = \frac{3}{2}, m_2 = \frac{3}{2}\right\rangle = \frac{3\hbar}{2},$$

$$\langle S = 3, m = 3|S_{1\,z}^2|S = 3, m = 3\rangle = \left\langle m_1 = \frac{3}{2}, m_2 = \frac{3}{2}\middle|S_{1\,z}^2\middle|m_1 = \frac{3}{2}, m_2 = \frac{3}{2}\right\rangle = \frac{9\hbar^2}{4},$$

$$\delta S_{1\,z}^2 = \frac{9\hbar^2}{4} - \frac{9\hbar^2}{4} = 0.$$

Nedoločenost tega stanja je 0. Stanje $|m_1 = 3/2, m_2 = 3/2\rangle$ je namreč lastno stanje operatorja H.
Za stanje $S = 3$, $m = 2$ za drugi spin dobimo:

$$\begin{aligned}\langle S = 3, m = 2|S_{2\,z}|S = 3, m = 2\rangle = \ & \tfrac{1}{2}\left\langle m_1 = \frac{3}{2}, m_2 = \frac{1}{2}\middle|S_{2\,z}\middle|m_1 = \frac{3}{2}, m_2 = \frac{1}{2}\right\rangle + \\ & \tfrac{1}{2}\left\langle m_1 = \frac{3}{2}, m_2 = \frac{1}{2}\middle|S_{2\,z}\middle|m_1 = \frac{1}{2}, m_2 = \frac{3}{2}\right\rangle + \\ & \tfrac{1}{2}\left\langle m_1 = \frac{1}{2}, m_2 = \frac{3}{2}\middle|S_{2\,z}\middle|m_1 = \frac{3}{2}, m_2 = \frac{1}{2}\right\rangle + \\ & \tfrac{1}{2}\left\langle m_1 = \frac{1}{2}, m_2 = \frac{3}{2}\middle|S_{2\,z}\middle|m_1 = \frac{1}{2}, m_2 = \frac{3}{2}\right\rangle =\end{aligned}$$

$$= \frac{1}{2}\frac{\hbar}{2} + 0 + 0 + \frac{1}{2}\frac{3\hbar}{2} = \hbar \quad \text{in na enak način}$$

$$\langle S = 3, m = 2|S_{2\,z}^2|S = 3, m = 2\rangle = \frac{1}{2}\frac{\hbar^2}{4} + 0 + 0 + \frac{1}{2}\frac{9\hbar^2}{4} = \frac{5\hbar^2}{4}, \quad \text{iz česar sledi}$$

$$\delta S_{2\,z}^2 = \frac{5\hbar^2}{4} - \hbar^2 = \frac{\hbar^2}{4}.$$