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Heisenbergovo načelo nedoločenosti

1. Izpelji Heizenbergovo načelo nedoločenosti za produkt nedoločenosti poljubnih hermitskih operatorev A in B.
2. Poišči valovno funkcijo z minimalnim produktom nedoločenosti lege in gibalne količine.

Rešitev:

Nekaj matematike

A in B hermitska $\Leftrightarrow A = A^+$ in $B = B^+$

Če je A hermitski, potem je tudi $A - \langle A \rangle$ hermitski.

Komutator je antihermitski

$$[A,B]^+ = (AB - BA)^+ = B^+A^+ - A^+B^+ = -(AB + BA) = -[A,B]$$

Antikomutator je hermitski

$$\{A,B\}^+ = (AB + BA)^+ = B^+A^+ + A^+B^+ = AB + BA = \{A,B\}$$

Pričakovane vrednosti hermitskega operatorja so realne

$$\langle A \rangle = \langle \Psi | A | \Psi \rangle = \langle A^+\Psi | \Psi \rangle = \langle A\Psi | \Psi \rangle = \langle \Psi | A\Psi \rangle^* = \langle A \rangle^*$$

Pričakovane vrednosti antihermitskega operatorja so imaginarne

$$\langle B \rangle = \langle \Psi | B | \Psi \rangle = \langle B^+\Psi | \Psi \rangle = \langle -B\Psi | \Psi \rangle = -\langle \Psi | B\Psi \rangle^* = -\langle B \rangle^*$$

Nedoločenosti poljubnih hermitskih operatorev A in B

$$\begin{aligned}
\delta^2 A \delta^2 B &= \left\langle (A - \langle A \rangle)^2 \right\rangle \left\langle (B - \langle B \rangle)^2 \right\rangle \\
&= \left\langle \Psi | (A - \langle A \rangle)^2 \Psi \right\rangle \left\langle \Psi | (B - \langle B \rangle)^2 \Psi \right\rangle \\
&= \langle (A - \langle A \rangle)\Psi | (A - \langle A \rangle)\Psi \rangle \langle (B - \langle B \rangle)\Psi | (B - \langle B \rangle)\Psi \rangle \\
&= \langle \Psi_1 | \Psi_1 \rangle \langle \Psi_2 | \Psi_2 \rangle
\end{aligned}$$

Schwartz-Cauchy-Bunjakovski

$$\langle \Psi_1 | \Psi_1 \rangle \langle \Psi_2 | \Psi_2 \rangle \geq |\langle \Psi_1 | \Psi_2 \rangle|^2$$

Enakost velja ko je $\Psi_2 = \lambda \Psi_1$ torej,

$$(A - \langle A \rangle)\Psi = \alpha(B - \langle B \rangle)\Psi \qquad \text{1.Pogoj}$$

kar je prvi pogoj za iskano funkcijo.

Neenakost Schwartz-Cauchy-Bunjakovski nam da,

$$\langle (A - \langle A \rangle)\Psi | (A - \langle A \rangle)\Psi \rangle \langle (B - \langle B \rangle)\Psi | (B - \langle B \rangle)\Psi \rangle \geq |\langle (A - \langle A \rangle)\Psi | (B - \langle B \rangle)\Psi \rangle|^2$$

$$|\langle (A-\langle A\rangle)\Psi|(B-\langle B\rangle)\Psi\rangle|^2 = |\langle\Psi|(A-\langle A\rangle)(B-\langle B\rangle)\Psi\rangle|^2$$
$$= \left|\left\langle\Psi|\frac{1}{2}([C,D]+\{C,D\})\Psi\right\rangle\right|^2$$
$$= \frac{1}{4}|\langle[C,D]\rangle+\langle\{C,D\}\rangle|^2$$
$$= \frac{1}{4}\left|\langle[C,D]\rangle^2+\langle\{C,D\}\rangle^2\right|$$
$$\geq \frac{1}{4}|\langle[C,D]\rangle|^2$$

Enakost velja, če je $\langle\{C,D\}\rangle = 0$

$$\langle\{A-\langle A\rangle, B-\langle B\rangle\}\rangle = 0 \qquad \text{2.Pogoj}$$

kar je drugi pogoj za iskano funkcijo.

Za nedoločenost dobimo

$$\delta A\delta B \geq \frac{1}{2}|\langle[A-\langle A\rangle, B-\langle B\rangle]\rangle| = \frac{1}{2}|\langle[A,B]\rangle|$$

Zgoraj smo uvedli $C = A-\langle A\rangle$ in $D = B-\langle B\rangle$, in zapisali $CD = \frac{CD+DC}{2} + \frac{CD-DC}{2} = \frac{1}{2}([C,D]+\{C,D\})$, prvi člen predstavlja komutator drugi člen pa antikomutator.

Iz 1. in 2. pogoja konstruirajmo funkcijo z minimalnim produktom nedoločenosti lege in gibalne količine

Prvi pogoj nam da

$$(x-\langle x\rangle)\Psi = \alpha(p-\langle p\rangle)\Psi$$
$$x\Psi-\langle x\rangle\Psi = -\alpha i\hbar\frac{\partial\Psi}{\partial x}-\alpha\langle p\rangle\Psi$$
$$\Psi\frac{-x+\langle x\rangle-\alpha\langle p\rangle}{\alpha i\hbar} = \frac{\partial\Psi}{\partial x}$$
$$\int\frac{d\Psi}{\Psi} = \frac{1}{\alpha i\hbar}\int(-x+\langle x\rangle-\alpha\langle p\rangle)dx$$
$$\ln\Psi = -\frac{1}{\alpha i\hbar}\left(\frac{x^2}{2}-\langle x\rangle x+\alpha\langle p\rangle x\right)+c_1$$
$$\ln\Psi = -\frac{1}{2\alpha i\hbar}\left((x-\langle x\rangle)^2-\langle x\rangle^2+2\alpha\langle p\rangle x\right)+c_1$$
$$\ln\Psi = -\frac{1}{2\alpha i\hbar}(x-\langle x\rangle)^2-\frac{\langle p\rangle x}{i\hbar}+c_2$$
$$\Psi = A\exp\left[\frac{i}{2\alpha\hbar}(x-\langle x\rangle)^2+\frac{\langle p\rangle xi}{\hbar}\right]$$

drugi pogoj

$$\langle\{x-\langle x\rangle, p-\langle p\rangle\}\rangle = 0$$
$$\langle\Psi|(x-\langle x\rangle)(p-\langle p\rangle) + (p-\langle p\rangle)(x-\langle x\rangle)|\Psi\rangle = 0$$
$$\langle(x-\langle x\rangle)\Psi|(p-\langle p\rangle)\Psi\rangle + \langle(p-\langle p\rangle)\Psi|(x-\langle x\rangle)\Psi\rangle = 0$$
$$\langle\alpha(p-\langle p\rangle)\Psi|(p-\langle p\rangle)\Psi\rangle + \langle(p-\langle p\rangle)\Psi|\alpha(p-\langle p\rangle)\Psi\rangle = 0$$
$$\alpha^*\left\langle(p-\langle p\rangle)^2\right\rangle + \alpha\left\langle(p-\langle p\rangle)^2\right\rangle = 0$$
$$\langle(p-\langle p\rangle)\rangle = 0 \Rightarrow \delta p = 0 \text{ nas ne zanima}$$
$$\alpha + \alpha^* = 0 \Rightarrow \alpha \text{ je imaginarn}$$

Definirajmo

$$\frac{i}{2\alpha\hbar} = -\frac{1}{4\sigma^2}$$

potem je

$$\Psi = A\exp\left[-\frac{1}{4\sigma^2}(x-\langle x\rangle)^2\right]\exp\left[i\frac{\langle p\rangle}{\hbar}x\right]$$
$$A = \frac{1}{\sqrt[4]{2\pi\sigma^2}}$$