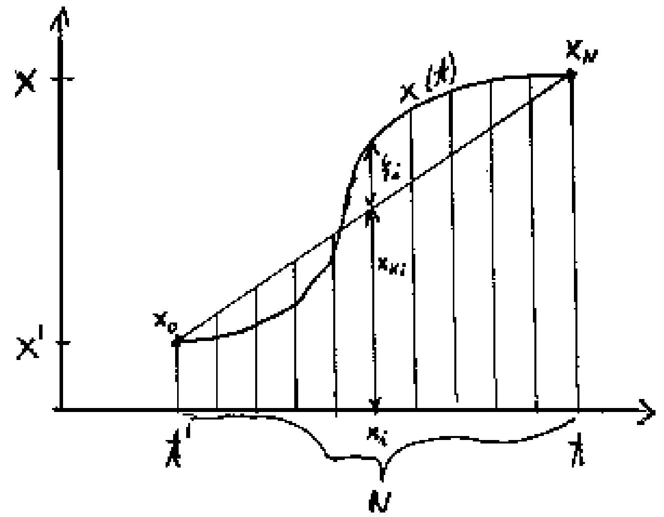
'PATH INTEGRAL' ZA PROST DELEC

$$K(x,t\,;x',t') = \int e^{\frac{i}{\hbar}S[x(t)]}\,\mathcal{D}x(t) \qquad ; \qquad V(x)=0$$

Najprej izračunamo akcijo za prost delec :



$$S[x(t)] = \int L(x(t))\,dt = \int \frac{1}{2}m\left(\frac{x_i-x_{i-1}}{t_i-t_{i-1}}\right)^2 dt$$

$$L = T-V = \frac{1}{2}mv^2 = \frac{1}{2}m\left(\frac{x_i-x_{i-1}}{t_i-t_{i-1}}\right)^2$$

To velja ker smo območje razdelili na dovolj majhne intervale, da je v posameznem intervalu hitrost podana $v=\frac{\Delta x}{\Delta t}$.
Za celotno akcijo seštejemo po vseh intervalih.

$$S[x(t)] = \sum_{i=1}^{N}\left[\frac{1}{2}m\left(\frac{x_i-x_{i-1}}{t_i-t_{i-1}}\right)\Delta t\right] = \sum_{i=1}^{N}\frac{m}{2}\frac{(x_i-x_{i-1})^2}{\Delta t}$$

Klasična limita za naš primer bi bila :

$$S_K = \int_{t'}^{t} L\,d\hat{t} = \int_{t'}^{t}\frac{1}{2}m\left(\frac{x-x'}{t-t'}\right)d\hat{t} = \frac{1}{2}m\left(\frac{x-x'}{t-t'}\right)^2(t-t') = \frac{1}{2}m\frac{(x-x')^2}{t-t'}$$

Vpeljemo nove oznake za koordinate x : $\quad x_i = x_{Ki}+\xi_i$
nadalje napišemo kot klasično razdaljo + razliko

$$S[x(t)] = \sum_{i=1}^{N}\frac{m}{2}\frac{(x_{Ki}+\xi_i-x_{K(i-1)}-\xi_{i-1})^2}{\Delta t} =$$

$$= \sum_{i=1}^{N}\frac{m}{2\Delta t}\left[(x_{Ki}-x_{K(i-1)})^2+(\xi_i-\xi_{i-1})^2+2(x_{Ki}-x_{K(i-1)})(\xi_i-\xi_{i-1})\right]$$

Pogledamo posamezne člene akcije:

$$-\sum_{i=1}^{N} \frac{m}{2\Delta t} \left(x_{Ki} - x_{K(i-1)}\right)^2 = \sum_{i=1}^{N} \frac{m \cdot N}{2(t-t')} \cdot \frac{(x-x')^2}{N^2} = N \frac{m}{2} \frac{(x-x')^2}{t-t'} \frac{1}{N} = S_K$$

$$\Delta t = \frac{t-t'}{N} \qquad \Delta x_K = \frac{x-x'}{N}$$

$$-\sum_{i=1}^{N} \frac{m 2}{2\Delta t} \left(\xi_i - \xi_{i-1}\right) \frac{(x-x')^2}{N^2} = 0$$

$$\sum_{i=1}^{N} (\xi_i - \xi_{i-1}) = (\cancel{\xi_1} - \xi_0) + (\cancel{\xi_2} - \cancel{\xi_1}) + (\cancel{\xi_3} - \cancel{\xi_2}) + \dots + (\xi_N - \cancel{\xi_{N-1}})$$

$$= \underset{0}{\underset{\parallel}{\xi_N}} - \underset{0}{\underset{\parallel}{\xi_0}} = 0$$

Za akcijo nam ostane izraz:

$$\boxed{S = S_K + \frac{m}{2\Delta t}\left[(\xi_1)^2 + (\xi_2 - \xi_1)^2 + \dots + (\xi_{N-1})^2\right]}$$

Sedaj lahko napišemo izraz za path integral za akcijo:

$$K(x,t;x',t') = A_N \cdot e^{\frac{i}{\hbar} S_K} \int d\xi_1 \int d\xi_2 \dots \int d\xi_N \, e^{\frac{i}{\hbar}\frac{m}{2\Delta t}\left[\xi_1^2 + (\xi_2 - \xi_1)^2 + \dots + \xi_{N-1}^2\right]}$$

Pri tem smo pred integral napisali konstanto A_N, ker smo pretvorili funkcionalni integral v navadnega in to zahtevajo mat. pravila. Izraz v eksponentu pretvorimo v popolne kvadrate:

$$\xi_1^2 + (\xi_2 - \xi_1)^2 = 2\xi_1^2 - 2\xi_1\xi_2 + \xi_2^2 = 2\left[\left(\xi_1 - \frac{\xi_2}{2}\right)^2 + \frac{\xi_2^2}{4}\right]$$

$$K = A_N e^{\frac{i}{\hbar} S_K} \int e^{\frac{i}{\hbar}\frac{m 2}{2\Delta t}\left[\left(\xi_1 - \frac{\xi_2}{2}\right)^2 + \frac{\xi_2^2}{4}\right]} d\xi_1 \int \dots$$

$$= A_N e^{\frac{i}{\hbar} S_K} \int e^{\frac{i}{\hbar}\frac{m}{\Delta t}\left(\xi_1 - \frac{\xi_2}{2}\right)^2} d\xi_1 \int e^{\frac{\xi_2^2}{4}} e^{\frac{im}{2\hbar\Delta t}(\xi_3 - \xi_2)^2} d\xi_2 \int \dots$$

Poglejmo rešitve posameznih integracij:

1) $\int e^{\frac{i}{\hbar}\frac{m}{\Delta t}(\xi_1-\frac{\xi_2}{2})^2}d\xi_1 =$ $u=\xi_1-\frac{\xi_2}{2}$

$du = d\xi_1$

$= \int e^{\frac{im}{\hbar\Delta t}u^2}du =$ $t = u\sqrt{\frac{-m}{\hbar\Delta t}}$

$$=\sqrt{\frac{\hbar\Delta t}{-m}}\underbrace{\int e^{-it^2}dt}_{\sqrt{i\pi}} = \sqrt{\frac{\pi\hbar\Delta t}{im}}$$

v naslednji integral

2) $\int e^{\frac{im}{\hbar\Delta t 2}[(\xi_3-\xi_2)^2+\frac{\xi_2}{2}]}d\xi_2 = \int e^{\frac{im}{2\hbar\Delta t}\frac{3}{2}(\xi_2-\frac{2}{3}\xi_3)^2}d\xi_2\, e^{\frac{im}{2\hbar\Delta t}\frac{1\cdot\xi_3^2}{3}}$

naredimo spet enake substitucije in dobimo

$$=\sqrt{\frac{2\pi\hbar\Delta t}{im}}\sqrt{\frac{2}{3}}$$

3) $\int e^{\frac{im}{\hbar\Delta t 2}[(\xi_4-\xi_3)^2+\frac{1}{3}\xi_3^2]}d\xi_3 = \int e^{\frac{im}{\hbar\Delta t 2}\frac{4}{3}(\xi_3-\frac{3}{4}\xi_4)^2}d\xi_3\cdot e^{\frac{im}{\hbar\Delta t 2}\frac{\xi_4^2}{4}}$

po izračunu dobimo:

$$=\sqrt{\frac{2\pi\hbar\Delta t}{im}}\cdot\sqrt{\frac{3}{4}}$$

Na koncu le še zmnožimo vse posamezne rešitve:

$$K = A_N e^{\frac{i}{\hbar}S_K}\cdot\sqrt{\frac{2\pi\hbar\Delta t}{im}}\frac{1}{\cancel{\sqrt{2}}}\cdot\sqrt{\frac{2\pi\hbar\Delta t}{im}}\frac{\cancel{\sqrt{2}}}{\cancel{\sqrt{3}}}\cdot\sqrt{\frac{2\pi\hbar\Delta t}{im}}\frac{\cancel{\sqrt{3}}}{\sqrt{4}}\cdots$$

$$\underline{\underline{K = A_N\frac{1}{\sqrt{N}}\left(\frac{2\pi\hbar\Delta t}{im}\right)^{\frac{N-1}{2}}e^{i\frac{m(x-x')^2}{2\hbar(t-t')}}}}$$

Ostane nam le še določitev konstante A_N.

$$\Psi(x,t) = \int K(x,t;x',t')\,\Psi(x',t')\,dx'$$

Gledamo za majhen $\Delta t \Rightarrow L$ je časovna konstanta

$$S = \int L\,dt = L\,\Delta t$$

$$\Psi(x,t+\Delta t) = \int\frac{1}{A}e^{\frac{i}{\hbar}L\Delta t}\,\Psi(y,t)\,dy \qquad L=\frac{mv^2}{2}$$

$$\Psi(x,t+\Delta t) = \int\frac{1}{A}e^{\frac{i}{\hbar}\Delta t\left(\frac{m}{2}\left(\frac{x-y}{\Delta t}\right)^2\right)}\,\Psi(y,t)\,dy$$

označimo : $y = x + \eta$

$$\Psi(x, t+\Delta t) = \int \frac{1}{A} e^{\frac{i}{\hbar}\Delta t\left(\frac{m}{2}\left(\frac{\eta}{\Delta t}\right)^2\right)} \Psi(x+\eta, t)\, d\eta$$

η je majhen, razvijemo :

$$\Psi(x,t) + \Delta t \frac{\partial \Psi}{\partial t} = \int \frac{1}{A} e^{\frac{i m \eta^2}{\hbar \Delta t\, 2}} \left[\Psi(x,t) + \eta \frac{d\Psi}{dt} \right] d\eta$$

Primerjamo le vodilna člena, ko gre $\Delta t \to 0$:

$$\Psi(x,t) = \Psi(x,t) \int \frac{1}{A} e^{\frac{i m \eta^2}{2\hbar \Delta t}} d\eta$$

$$u = \sqrt{\frac{-m}{\hbar \Delta t\, 2}}\ \eta$$

$$du = \sqrt{\frac{-m}{2\hbar \Delta t}}\ d\eta$$

$$A = \int \sqrt{\frac{2\hbar \Delta t}{i^2 m}}\ e^{-iu^2} du = \sqrt{\frac{2\hbar \Delta t \cdot i\pi}{i^2 m}}$$

$$A = \sqrt{\frac{2\pi\hbar \Delta t}{i m}}$$

$$A_N = \left(\frac{1}{A}\right)^N = \left(\frac{2\pi\hbar\Delta t}{im}\right)^{-\frac{N}{2}}$$

Za končni rezultat tako dobimo :

$$\boxed{K(x,t\,;\,x',t') = \sqrt{\frac{i m}{2\pi\hbar(t-t')}}\ e^{\frac{im(x-x')^2}{2\hbar(t-t')}}}$$