Harmonski oscilator

Aleš Česen

5. april 2004

Za harmonski oscilator velja:

$$H = \frac{p^2}{2m} + \frac{1}{2}kx^2$$

$$H = \hbar\omega\left(a^\dagger\, a + \frac{1}{2}\right) \tag{1}$$

$$\omega = \sqrt{\frac{k}{m}}$$

Zapišimo še anihalicijski operator a in kreacijski operator $a^\dagger$:

$$a = \frac{1}{\sqrt{2}}\left(\frac{x}{x_0} + i\frac{p}{p_0}\right) \tag{2}$$

$$a^\dagger = \frac{1}{\sqrt{2}}\left(\frac{x}{x_0} - i\frac{p}{p_0}\right) \tag{3}$$

$$\left[a, a^\dagger\right] = 1 \tag{4}$$

Kjer sem označil:

$$x_0 = \sqrt{\tfrac{\hbar}{m\omega}} \qquad \text{in} \qquad p_0 = \tfrac{\hbar}{x_0}$$

$$E_n = \hbar\omega\left(n + \frac{1}{2}\right) \tag{5}$$

1 Zapis lastnih stanj

$$a^\dagger\,|n\rangle = \sqrt{n+1}\,|n+1\rangle \tag{6}$$

$$a\,|n\rangle = \sqrt{n}\,|n-1\rangle \tag{7}$$

$$|n\rangle = \frac{\left(a^\dagger\right)^n}{\sqrt{n!}}\,|0\rangle \tag{8}$$

2 Časovni razvoj

Na začetku imamo funkcijo:

$$|\Psi, 0\rangle = \frac{1}{\sqrt{2}}\left(|0\rangle + |1\rangle\right) \tag{9}$$

Hitro dobimo časovni razvoj te funkcije, saj imamo opravka z lastnimi stanji:

$$\begin{aligned} |\Psi, t\rangle &= \frac{1}{\sqrt{2}}\left(|0\rangle\, e^{-i\frac{E_0}{\hbar}t} + |1\rangle\, e^{-i\frac{E_1}{\hbar}t}\right) \\ &= \frac{1}{\sqrt{2}}\left(|0\rangle\, e^{-i\frac{\omega}{2}t} + |1\rangle\, e^{-i\frac{3\omega}{2}t}\right) \end{aligned} \tag{10}$$

Tu upoštevamo enačbo (5).

3 Pričakovane vrednosti

Poglejmo si, kako je s pričakovanimi vrednostmi za splošen $\langle A\rangle$:

$$\begin{aligned} \langle A\rangle &= \langle\Psi\,|A|\,\Psi\rangle = \left(\frac{1}{\sqrt{2}}\right)^2 \langle 0\,|A|\,0\rangle + \left(\frac{1}{\sqrt{2}}\right)^2 \langle 1\,|A|\,1\rangle + \\ &\quad + \frac{1}{2}\langle 0\,|A|\,1\rangle\, e^{i\frac{\omega}{2}t} e^{-i\frac{3\omega}{2}t} + \frac{1}{2}\langle 1\,|A|\,0\rangle\, e^{i\frac{3\omega}{2}t} e^{-i\frac{\omega}{2}t} \\ &= \frac{1}{2}\langle 0\,|A|\,0\rangle + \frac{1}{2}\langle 1\,|A|\,1\rangle + Re\left(\langle 1\,|A|\,0\rangle\, e^{i\omega t}\right) \end{aligned} \tag{11}$$

3.1 Pričakovana vrednost koordinate

$$A = x$$

Hitro ugotovimo, da velja $\langle 0\,|x|\,0\rangle = \langle 1\,|x|\,1\rangle = 0$. Tako nam od enačbe (11) ostane le še:

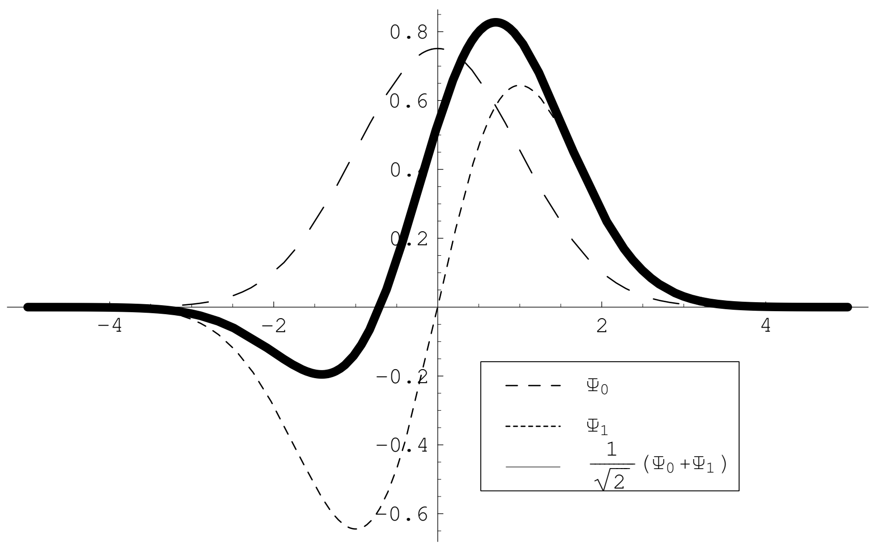
$$\langle x\rangle = Re\left(\langle 1\,|x|\,0\rangle\, e^{i\omega t}\right) \tag{12}$$

Za lažje računanje x izrazimo z a in $a^\dagger$. Enačbi (2) in (3) seštejem in dobim:

$$\begin{aligned} a + a^\dagger &= \frac{2}{\sqrt{2}}\frac{x}{x_0} \\ x &= \frac{\sqrt{2}}{2}x_0\left(a + a^\dagger\right) \end{aligned} \tag{13}$$

$$
\begin{aligned}
\langle x\rangle &= Re\left(\left\langle 1\left|\frac{\sqrt{2}}{2}x_0\left(a+a^\dagger\right)\right|0\right\rangle e^{i\omega t}\right)\\
&= \frac{\sqrt{2}x_0}{2}Re\left[\left(\overbrace{\langle 1\left|a\right|0\rangle}^{0}+\left\langle 1\left|a^\dagger\right|0\right\rangle\right)e^{i\omega t}\right]\\
&= \frac{\sqrt{2}x_0}{2}Re\left(\left\langle 1\left|\sqrt{1}\right|1\right\rangle e^{i\omega t}\right)\\
\langle x\rangle &= \frac{\sqrt{2}}{2}x_0\cos(\omega t) \qquad (14)
\end{aligned}
$$

Tu smo v drugi vrstici računa upoštevali ortonormiranost stanj: $\langle i|j\rangle = \delta_{i,j}$.
Za boljšo predstavo si poglejmo, kako izgledata fukciji osnovnega $\Psi_0(x)$ in prvega vzbujenega $\Psi_1(x)$ stanja, ter kakšna je naša fukcija $\Psi(x) = \frac{1}{\sqrt{2}}\left(\Psi_0(x) + \Psi_1(x)\right)$, ki je linearna kombinacija prvih dveh. Vse seveda za čas $t = 0$:



3.2 Pričakovana vrednost gibalne količine

$$A = p$$

Po enačbi (11) dobimo za $A = p$:

$$
\begin{aligned}
\langle p\rangle &= \frac{1}{2}\left(\overbrace{\langle 0\left|p\right|0\rangle}^{0}+\overbrace{\langle 1\left|p\right|1\rangle}^{0}\right)+Re\left(\langle 1\left|p\right|0\rangle e^{i\omega t}\right)\\
\langle p\rangle &= Re\left(\langle 1\left|p\right|0\rangle e^{i\omega t}\right) \qquad (15)
\end{aligned}
$$

Podobno kot pri x , sedaj p izrazim z a in $a^\dagger$. Enačbi (2) in (3) odštejem in dobim:

$$\begin{aligned} a - a^\dagger &= \frac{2}{\sqrt{2}} i \frac{p}{p_0} \\ p &= i\frac{\sqrt{2}}{2} p_0 \left(a^\dagger - a\right) \end{aligned} \quad (16)$$

$$\begin{aligned} \langle p\rangle &= Re\left(\left\langle 1 \left| i\frac{\sqrt{2}}{2} p_0 \left(a^\dagger - a\right) \right| 0 \right\rangle e^{i\omega t}\right) \\ &= \frac{\sqrt{2}}{2} p_0 Re\left[\left\langle 1 \left| i\left(a^\dagger - \overbrace{a}^{\longrightarrow 0}\right) \right| 0 \right\rangle + e^{i\omega t}\right] \\ &= \frac{\sqrt{2}}{2} p_0 Re\left(\left\langle 1 \left| \sqrt{1} \right| 1 \right\rangle i e^{i\omega t}\right) \\ \langle p\rangle &= -\frac{p_0}{\sqrt{2}} \sin(\omega t) \end{aligned} \quad (17)$$

3.3 Pričakovana vrednost kvadrata koordinate

$$A = x^2$$

Iz enačbe (13) dobim enačbo za x^2:

$$x^2 = \frac{x_0^2}{2}\left(a^2 + {a^\dagger}^2 + a\,a^\dagger + a^\dagger\,a\right) \quad (18)$$

$$\begin{aligned} \left\langle x^2 \right\rangle &= \frac{x_0^2}{2}\left(\frac{1}{2}\left\langle 0 \left| a\,a^\dagger \right| 0 \right\rangle + \frac{1}{2}\left\langle 1 \left| a\,a^\dagger + a^\dagger\,a \right| 1 \right\rangle + Re(0)\right) \\ &= \frac{x_0^2}{4}\left(\langle 0 |a| 1\rangle + \sqrt{2}\langle 1 |a| 2\rangle + \left\langle 1 \left| a^\dagger \right| 0 \right\rangle\right) \\ &= \frac{x_0^2}{4}\left(1 + \sqrt{2}\sqrt{2} + 1\right) \\ \left\langle x^2 \right\rangle &= x_0^2 \end{aligned} \quad (19)$$

4 Verjetnost

Sedaj bomo izračunali verjetnost $P_{x<0}$, da se delec nahaja na levi polovici potenciala.

$$P_{x<0} = \int_{-\infty}^{0} \left|\Psi(x,t)\right|^2 dx \quad (20)$$

$$\Psi(x,t) = \frac{1}{\sqrt{2}}\left(\Psi_0(x,t) + \Psi_1(x,t)\right) \tag{21}$$

Zapišemo znani lastni fukciji harmonskega oscilatorja $\Psi_0(x,t)$ in $\Psi_1(x,t)$:

$$\Psi_0(x,t) = \frac{1}{\sqrt[4]{\pi\, x_0^2}}\, e^{-\frac{x^2}{2\,x_0^2}}\, e^{-\frac{i}{2}\,t\,\omega} \tag{22}$$

$$\Psi_1(x,t) = \frac{x\,\sqrt{2}}{x_0}\frac{1}{\sqrt[4]{\pi\, x_0^2}}\, e^{-\frac{x^2}{2\,x_0^2}}\, e^{-\frac{3\,i}{2}\,t\,\omega} \tag{23}$$

Iz enačbe (20) dobim:

$$P_{x<0} = \frac{1}{2} - \frac{\cos(\omega\, t)}{\sqrt{2\,\pi}} \tag{24}$$

Ta časovni potek verjetnosti, da se delec nahaja na levi strani potenciala, narišem na ustrezen graf, kjer pišem φ kot fazo $\varphi = t\frac{\omega}{2\pi}$: