HEISENBERGOV PRINCIP NEDOLOČENOSTI

NALOGA:

a.) izpelji Heisenbergovo načelo nedoločenosti za produkt nedoločenosti poljubnih hermitskih operatorjev A in B

b.) poišči valovno funkcijo z minimalnim produktom nedoločenosti lege in gibalne količine

REŠITEV:

$\delta A \, \delta B = ?$, kjer sta A in B hermitska

$\Rightarrow A = A^+ \; ; \; B = B^+$

$\delta^2 A = \langle A^2 \rangle - \langle A \rangle^2 = \langle (A - \langle A \rangle)^2 \rangle \rightarrow$ spet hermitski

$$\delta^2 A \, \delta^2 B = \langle (A - \langle A \rangle)^2 \rangle \langle (B - \langle B \rangle)^2 \rangle =$$
$$= \langle \Psi | (A - \langle A \rangle)^2 \Psi \rangle \langle \Psi | (B - \langle B \rangle)^2 \Psi \rangle =$$
$$= \langle \underbrace{(A - \langle A \rangle)\Psi}_{\Psi_1} | \underbrace{(A - \langle A \rangle)\Psi}_{\Psi_1} \rangle \langle \underbrace{(B - \langle B \rangle)\Psi}_{\Psi_2} | \underbrace{(B - \langle B \rangle)\Psi}_{\Psi_2} \rangle \geq \quad ①$$
$$\geq \left| \langle (A - \langle A \rangle)\Psi | (B - \langle B \rangle)\Psi \rangle \right|^2 =$$
$$= \left| \langle \Psi | \underbrace{(A - \langle A \rangle)}_{C} \underbrace{(B - \langle B \rangle)}_{D} | \Psi \rangle \right|^2 =$$
$$= \left| \langle \Psi | \frac{[C,D] + \{C,D\}}{2} \Psi \rangle \right|^2 =$$
$$= \frac{1}{4} \left| \langle [C,D] \rangle + \langle \{C,D\} \rangle \right|^2$$

$\langle \Psi | A^2 \Psi \rangle = \langle A\Psi | A\Psi \rangle$

neenakost Schwarz-Cauchy-Bunjakovski

$\langle \Psi_1 | \Psi_1 \rangle \langle \Psi_2 | \Psi_2 \rangle \geq |\langle \Psi_1 | \Psi_2 \rangle|^2$

enakost je izpolnjena pri $\Psi_2 = \lambda \Psi_1$

① enakost velja

$(A - \langle A \rangle)\Psi = \alpha (B - \langle B \rangle)\Psi$

to je prvi pogoj za iskano valovno funkcijo

velja

$$CD = \frac{CD + DC + CD - DC}{2} =$$
$$= \frac{1}{2}\left([C,D] + \{C,D\}\right)$$

$[C,D]$ komutator

$\{C,D\}$ antikomutator

Pred nadaljevanjem pokažimo nekaj matematičnih lastnosti

hermitski operator <=> $A^+ = A$

antihermitski operator <=> $B^+ = -B$

komutator: def: $[A,B] = AB - BA$ je antihermitski

dokaz:

$$[A,B]^+ = (AB - BA)^+ = (AB)^+ - (BA)^+ = B^+A^+ - A^+B^+ =$$
$$= -(AB - BA) = -[A,B]$$

antikomutator: def: $\{A,B\} = AB + BA$ je hermitski

dokaz:

$$\{A,B\}^+ = (AB + BA)^+ = (AB)^+ + (BA)^+ = B^+A^+ + A^+B^+ =$$
$$= AB + BA = \{A,B\}$$

Pričakovane vrednosti hermitskega operatorja so realne.

dokaz:

$$\langle A\rangle = \langle\Psi|A|\Psi\rangle = \langle A^+\Psi|\Psi\rangle = \langle A\Psi|\Psi\rangle = \langle\Psi|A\Psi\rangle^* = \langle A\rangle^*$$

Pričakovane vrednosti antihermitskega operatorja so imaginarne.

dokaz:

$$\langle B\rangle = \langle\Psi|B|\Psi\rangle = \langle B^+\Psi|\Psi\rangle = \langle -B\Psi|\Psi\rangle = \langle\Psi|B\Psi\rangle^* = -\langle B\rangle^*$$

nadaljujemo z reševanjem

$$\frac{1}{4}|\langle[C,D]\rangle + \langle\{C,D\}\rangle|^2 = \frac{1}{4}\left(\langle\{C,D\}\rangle^2 + |\langle[C,D]\rangle|^2\right) \geq$$

$$\geq \frac{1}{4}|\langle[C,D]\rangle|^2$$

enakost velja, če je $\langle\{C,D\}\rangle^2 = 0$ ②

to je drugi pogoj za iskano valovno funkcijo

$$\delta A \delta B \geq \frac{1}{2}|\langle[A-\langle A\rangle, B-\langle B\rangle]\rangle| = \frac{1}{2}|\langle[A,B]\rangle|$$

$\langle A\rangle$ in $\langle B\rangle$ sta skalarja in zato komutirata z operatorjem

$$\delta x \delta p \geq \frac{1}{2}|\langle[x,p]\rangle| = \frac{1}{2}|\langle i\hbar\rangle| = \frac{\hbar}{2} \Rightarrow \underline{\underline{\delta x \delta p \geq \frac{\hbar}{2}}}$$

Konstrukcija valovne funkcije iz pogojev ① in ②

① $(x-\langle x\rangle)\Psi = \lambda(p-\langle p\rangle)\Psi$

$$x\Psi - \langle x\rangle\Psi = -\lambda i\hbar\frac{\partial\Psi}{\partial x} - \lambda\langle p\rangle\Psi$$

$$\frac{\partial\Psi}{\partial x} + \frac{x}{\lambda i\hbar}\Psi + \Psi\,\frac{(\lambda\langle p\rangle - \langle x\rangle)}{\lambda i\hbar} = 0$$

$$\int\frac{d\Psi}{\Psi} = \int\left(-x + \langle x\rangle - \lambda\langle p\rangle\right)\frac{1}{\lambda i\hbar}\,dx$$

$$\ln\Psi = -\frac{1}{\lambda i\hbar}\left(\frac{x^2}{2} + (\lambda\langle p\rangle - \langle x\rangle)x\right) + \text{konst}$$

$$\ln\Psi = -\frac{1}{2i\hbar\lambda}\left((x-\langle x\rangle)^2 - \langle x\rangle^2 + \lambda\langle p\rangle x\right) + \text{konst}$$

$$\ln\Psi = -\frac{1}{2i\hbar\lambda}(x-\langle x\rangle)^2 - \frac{\langle p\rangle}{i\hbar}x + \text{konst}$$

$$\Psi = A e^{\frac{i}{2\alpha\hbar}(x-\langle x\rangle)^2} e^{i\frac{\langle p\rangle}{\hbar}x}$$

② $\langle \{(x-\langle x\rangle), (p-\langle p\rangle)\}\rangle = 0$

$$\langle \Psi | (x-\langle x\rangle)(p-\langle p\rangle) + (p-\langle p\rangle)(x-\langle x\rangle) | \Psi\rangle = 0$$

$$\langle (x-\langle x\rangle)\Psi | (p-\langle p\rangle)\Psi\rangle + \langle (p-\langle p\rangle)\Psi | (x-\langle x\rangle)\Psi\rangle = 0$$

$$\langle \alpha(p-\langle p\rangle)\Psi | (p-\langle p\rangle)\Psi\rangle + \langle (p-\langle p\rangle)\Psi | \alpha(p-\langle p\rangle)\Psi\rangle = 0$$

$$\alpha^* \langle (p-\langle p\rangle)^2\rangle + \alpha\langle (p-\langle p\rangle)^2\rangle = 0$$

$$(\alpha+\alpha^*)\ \langle (p-\langle p\rangle)^2\rangle = 0$$

I. $\langle (p-\langle p\rangle)^2\rangle = 0 \Rightarrow \delta p = 0 \rightarrow\!\leftarrow$

II. $\alpha + \alpha^* = 0$

$\alpha^* = -\alpha \Rightarrow \alpha$ je imaginaren

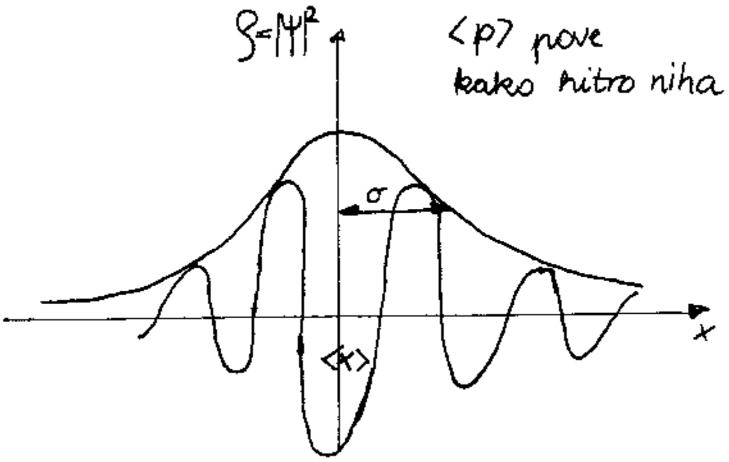
definirajmo σ:

$$\frac{i}{2\alpha\hbar} = -\frac{1}{4\sigma^2}$$

potem je

$$\boxed{\Psi = A e^{-\frac{1}{4\sigma^2}(x-\langle x\rangle)^2} e^{i\frac{\langle p\rangle}{\hbar}x}}$$

to je Gaussov valovni paket



Normalizacijo $A = \frac{1}{\sqrt[4]{2\pi\sigma^2}}$ dobimo tako, da Ψ primerjamo z normirano Gaussovo funkcijo