Valovni paket II

Elektron je ob času t=0 v stanju z valovno funkcijo $\psi(x,0)=\frac{1}{\sqrt[4]{2\pi\sigma_0^2}}e^{-\frac{x^2}{4\sigma_0^2}}e^{ik_0x}$.

Izračunaj časovni rarvoj valovne funkcije , če se elektron giblje v prostoru, kjer je potencial konstanten. Kako se s časom spreminja verjetnostna gostota?

$\langle x\rangle=0$, $\sigma_x=\sigma_0$, $\langle p\rangle=\hbar k_0$

$\psi(x,t)=?$

Valovno funkcijo razvijemo po lastnih funkcijah $\psi_k=e^{ikx}$.

Za lastne energije pa dobimo $E_k=\frac{\hbar^2k^2}{2m}$.

Lastnih funkcij $\psi_k=e^{ikx}$ ne moremo normirati, vidimo pa, da velja:

$$\psi(x)=\sum_k c_k\psi_k=\int c(k)\psi_k\,dk=\int c(k)e^{ikx}dk\ ,$$

kar je Fourierova transformacija.

Ker so v valovnem paketu zastopani vsi k, vsota preide v integral.

$$c(k)=\langle\psi_k|\psi\rangle=\int e^{-ikx}\psi\,dx=\int e^{-ikx}\frac{1}{\sqrt[4]{2\pi\sigma_0^2}}e^{-\frac{x^2}{4\sigma_0^2}}e^{ik_0x}dx$$

Opazimo, da je to ravno obratna Fourierova transformacija, zato dodamo še $\frac{1}{2\pi}$:

$$c(k)=\frac{1}{2\pi}\int e^{-ikx}\frac{1}{\sqrt[4]{2\pi\sigma_0^2}}e^{-\frac{x^2}{4\sigma_0^2}}e^{ik_0x}dx=\frac{1}{2\pi}\frac{1}{\sqrt[4]{2\pi\sigma_0^2}}\int e^{-ix(k-k_0)-\frac{x^2}{4\sigma_0^2}}dx$$

Eksponent prevedemo na popoln kvadrat:

$$-ix(k-k_0)-\frac{x^2}{4\sigma_0^2}=-\left[ix(k-k_0)+\frac{x^2}{4\sigma_0^2}+\left(i(k-k_0)\sigma_0\right)^2-\left(i(k-k_0)\sigma_0\right)^2\right]=-\left(i(k-k_0)\sigma_0+\frac{x}{2\sigma_0}\right)^2-(k-k_0)^2\sigma_0^{\,2}$$

$$c(k)=\frac{1}{2\pi}\frac{1}{\sqrt[4]{2\pi\sigma_0^2}}\int e^{-\left(i(k-k_0)\sigma_0+\frac{x}{2\sigma_0}\right)^2-(k-k_0)^2\sigma_0^2}dx=\frac{1}{2\pi}\frac{e^{-(k-k_0)^2\sigma_0^2}}{\sqrt[4]{2\pi\sigma_0^2}}\int e^{-\left(i(k-k_0)\sigma_0+\frac{x}{2\sigma_0}\right)^2}dx=$$

$$=\frac{1}{2\pi}\frac{e^{-(k-k_0)^2\sigma_0^2}}{\sqrt[4]{2\pi\sigma_0^2}}\int e^{-u^2}2\sigma_0du=\frac{\sigma_0}{\pi}\frac{e^{-(k-k_0)^2\sigma_0^2}}{\sqrt[4]{2\pi\sigma_0^2}}\sqrt{\pi}$$

Vpeljala sem novo spremenljivko u.

$u=i(k-k_0)\sigma_0+\frac{x}{2\sigma_0}$, $du=\frac{dx}{2\sigma_0}$

Za reševanje integralov zgornjega tipa uporabimo nastavek:

$$\int_{-\infty}^{\infty} e^{-ak^2} dk = \sqrt{\frac{\pi}{a}}\,, \qquad \mathrm{Re}(a) \succ 0$$

$$\psi(x,t) = \int c(k) e^{ikx} e^{-i\omega_k t} dk = \frac{\sigma_0}{\sqrt{\pi}} \frac{1}{\sqrt[4]{2\pi\sigma_0^2}} \int e^{-(k-k_0)^2 \sigma_0^2} e^{ikx} e^{-i\frac{k^2\hbar}{2m}t} dk$$

Eksponent spravimo na popoln kvadrat:

$$-\left(k^2 - 2kk_0 + k_0^{\ 2}\right)\sigma_0^{\ 2} + ikx - i\frac{k^2\hbar}{2m}t = -\left(\sigma_0^{\ 2} + i\frac{\hbar}{2m}t\right)\left(k - \frac{k_0\sigma_0^{\ 2} + \frac{i}{2}x}{\sigma_0^{\ 2} + i\frac{\hbar}{2m}t}\right)^2 + \frac{k_0\sigma_0^{\ 2} + \frac{i}{2}x}{\sigma_0^{\ 2} + i\frac{\hbar}{2m}t} - k_0^{\ 2}\sigma_0^{\ 2}$$

In na koncu dobimo:

$$\psi(x,t) = \frac{\sigma_0}{\sqrt{\pi}} \frac{\sqrt{\pi}}{\sqrt[4]{2\pi\sigma_0^2}} e^{\frac{\left(k_0\sigma_0^{\ 2} + \frac{i}{2}x\right)^2}{\sigma_0^{\ 2} + i\frac{\hbar}{2m}t} - k_0^{\ 2}\sigma_0^{\ 2}} \sqrt{\frac{1}{\sigma_0^{\ 2} + i\frac{\hbar}{2m}t}}$$

Zanima nas tudi kako se s časom spreminja verjetnostna gostota.

$$\rho(x,t) = \psi^*\psi = a^* a \cdot e^{b^*+b} = |a|^2 e^{2\mathrm{Re}\, b}; \qquad \psi = a \cdot e^b$$

$$\rho(x,t) = \frac{1}{\sqrt{2\pi\sigma_0^2}} \frac{1}{\sqrt{1 + \left(\frac{\hbar t}{2m}\right)^2}} e^{-\frac{\left(x - \sigma_0 \frac{\hbar t}{2m}\right)^2}{2\sigma_0^{\ 2}\left(1 - \frac{\hbar t}{2m\sigma_0^{\ 2}}\right)}}$$

Vpeljemo:

$$\sigma(t) = \sigma \sqrt{1 + \left(\frac{\hbar t}{m\sigma_0^{\ 2}}\right)^2}\ ; \quad v = \frac{p}{m} = \frac{\hbar k_0}{m}$$

in za verjetnostno gostoto dobimo:

$$\rho(x,t) = \frac{1}{\sqrt{2\pi\sigma(t)^2}} e^{-\frac{(x-vt)^2}{2\sigma(t)^2}}$$