# Casovni razvoj sestavljene funkcije delca v neskoncni potencialni jami

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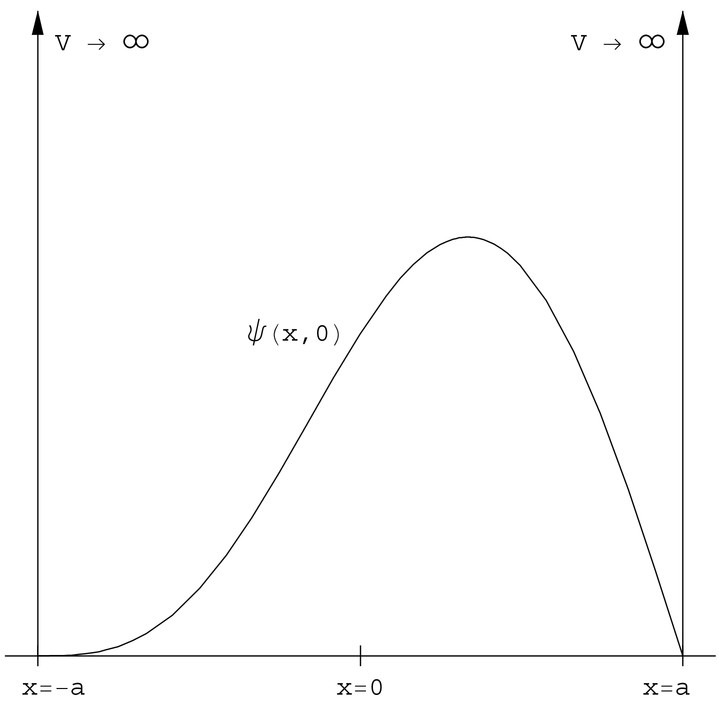
## Naloga

Imamo neskoncno potencialno jamo sirine 2a in v njej delec, ki ga ob casu t=0 opisemo s funkcijo:

$$\psi(x, 0) = A\left[\cos\frac{\pi x}{2a} + \frac{1}{2}\sin\frac{2\pi x}{2a}\right]$$




Zanima nas:

- kaksen je casovni razvoj funkcije
- verjetnost, da je delec na desni polovice jame
- pricakovana vrednost koordinate
- pricakovana vrednost velikosti gibalne kolicine

Lastne energije v neskoncni potencialni jami:

$$i\hbar \frac{\partial}{\partial t} \Psi(x, t) = H \Psi(x, t)$$

$$H \Psi_n(x) = E_n \Psi_n(x) \qquad E_n = \frac{\hbar^2 n^2 \pi^2}{2 (2 a)^2}$$

Razvoj po lastnih funkcijah in casovni razvoj lastnih funkcij:

$$\Psi(x, 0) = \sum_n c_n \Psi_n(x) \quad c_n = \int_{-\infty}^{\infty} \Psi_n^*(x, 0) \Psi(x, 0) \, dx$$

$$\Psi(x, t) = \sum_n c_n \Psi_n(x) e^{-\frac{i}{\hbar} E_n t}$$

Pricakovana vrednost operatorja:

$$< x > = \int \psi^* x \psi \, dx$$

$$< \hat{p} > = \int -\psi^* i\hbar \frac{\partial}{\partial x} \psi \, dx \quad \text{ali} \quad < \hat{p} > = m \frac{d < x >}{dt}$$

# Casovni razvoj

normalizacija

$$\int_{-a}^{a} \Psi^* \Psi \, dx = 1 = A^2 \int_{-a}^{a} \left(\cos \frac{\pi x}{2 a} + \frac{1}{2} \sin \frac{2 \pi x}{2 a}\right)^2 dx =$$

$$A^2 \int_{-a}^{a} \left[\cos^2 \frac{\pi x}{2 a} + \frac{1}{4} \sin^2 \frac{2 \pi x}{2 a} + \sin \frac{2 \pi x}{2 a} \cos \frac{\pi x}{2 a}\right] dx =$$

$$A^2 \left(a + \frac{a}{4}\right) = A^2 \frac{5 a}{4} = 1$$

$$\mathbf{A = \sqrt{\frac{4}{5 a}}}$$

razvoj po lastnih funkcijah

$$\Psi(x, 0) = 2 \sqrt{\frac{1}{5 a}} \cos \frac{\pi x}{2 a} + \sqrt{\frac{1}{5 a}} \sin \frac{2 \pi x}{2 a}$$

$$\Psi_1(x) = \frac{1}{\sqrt{a}} \cos \frac{\pi x}{2 a} \qquad \Psi_2(x) = \frac{1}{\sqrt{a}} \sin \frac{2 \pi x}{2 a}$$

$$\mathbf{c_1 = \frac{2}{\sqrt{5}} \qquad c_2 = \frac{1}{\sqrt{5}} \qquad c_n = 0;\ n > 2}$$

casovni razvoj

$$\Psi(x,t) = \frac{2}{\sqrt{5a}} \cos\frac{\pi x}{2a}\, e^{-\frac{i}{\hbar}E_1 t} + \frac{1}{\sqrt{5a}} \sin\frac{2\pi x}{2a}\, e^{-\frac{i}{\hbar}E_2 t}$$

$$\boldsymbol{\Psi(x,t) = \frac{1}{\sqrt{5a}}\, e^{-\frac{i}{\hbar}E_1 t}\left(2\cos\frac{\pi x}{2a} + \sin\frac{2\pi x}{2a}\, e^{-i\omega t}\right)}$$

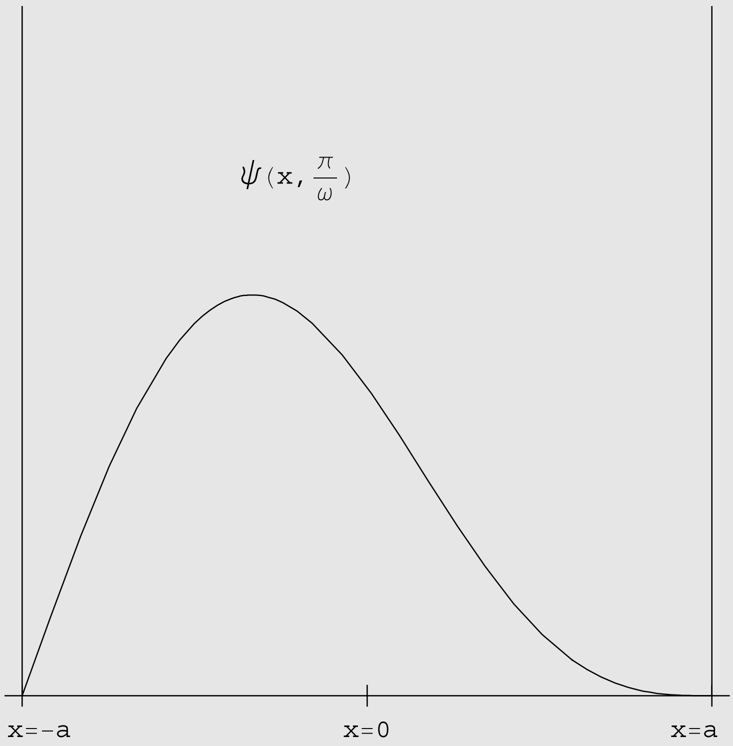
$$E_1 = \frac{\hbar^2\pi^2}{2(2a)^2} \qquad E_2 = \frac{2\hbar^2\pi^2}{(2a)^2}$$

$$\omega = \frac{E_2 - E_1}{\hbar}$$

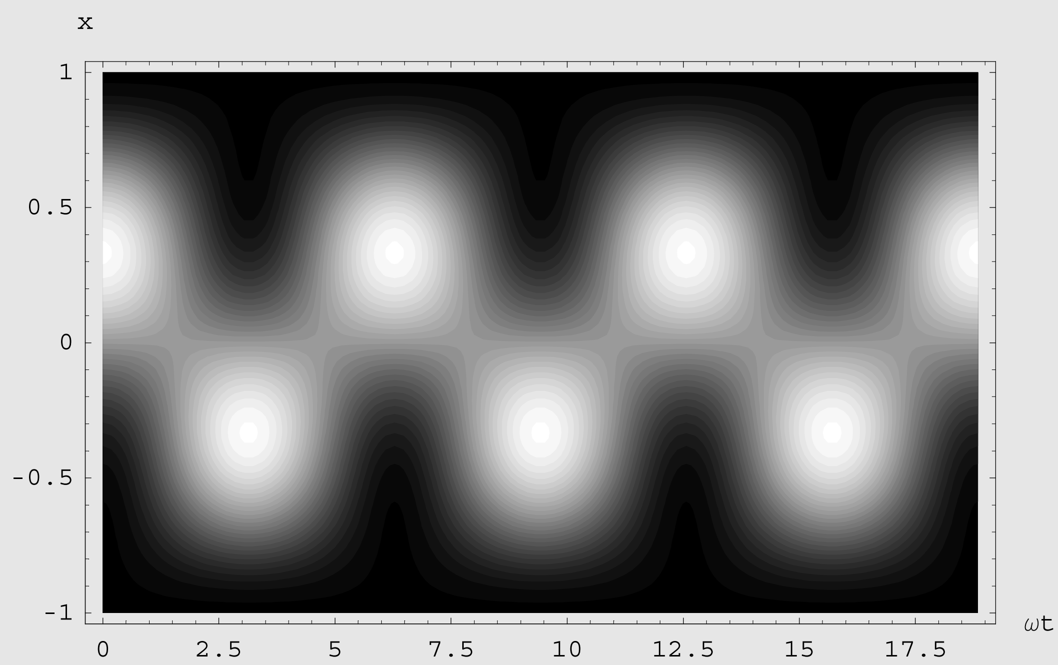
## ■ Valovna funkcija ob casu wt=p (pol periode)

$$\Psi\left(x, \frac{\pi}{\omega}\right) = \frac{1}{\sqrt{5a}}\, e^{-\frac{i}{\hbar}E_1\frac{\pi}{\omega}}\left(2\cos\frac{\pi x}{2a} + \sin\frac{2\pi x}{2a}\, e^{-i\pi}\right)$$

$$\boldsymbol{\Psi\left(x, \frac{\pi}{\omega}\right) \propto \frac{1}{\sqrt{5a}}\left(2\cos\frac{\pi x}{2a} - \sin\left(\frac{2\pi x}{2a}\right)\right)}$$



verjetnostna gostota ($\psi^* \psi$) v odvisnosti od casa :




# Verjetnost, da je elektron na desni polovici jame

$$P(t) = \int_0^a \Psi^*(x,t)\,\Psi(x,t)\,dx =$$

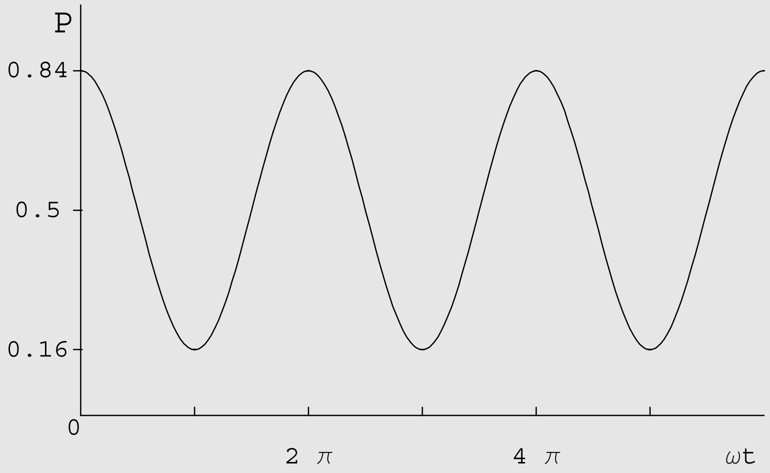
$$\int_0^a \left(\frac{2}{\sqrt{5}}\Psi_1^* e^{\frac{i}{\hbar}E_1 t} + \frac{1}{\sqrt{5}}\Psi_2^* e^{\frac{i}{\hbar}E_2 t}\right)\left(\frac{2}{\sqrt{5}}\Psi_1 e^{-\frac{i}{\hbar}E_1 t} + \frac{1}{\sqrt{5}}\Psi_2 e^{-\frac{i}{\hbar}E_2 t}\right) dx =$$

$$\int_0^a \left(\frac{4}{5}\Psi_1^*\Psi_1 + \frac{1}{5}\Psi_2^*\Psi_2 + \frac{2}{5}\Psi_2^*\Psi_1 e^{\frac{i}{\hbar}(E_2-E_1)t} + \frac{2}{5}\Psi_2\Psi_1^* e^{\frac{i}{\hbar}(E_1-E_2)t}\right) dx =$$

$$\frac{4}{10} + \frac{1}{10} + \frac{2}{5}\int_0^a \frac{1}{a}\cos\frac{\pi x}{2a}\sin\frac{2\pi x}{2a}\left(e^{\frac{i}{\hbar}(E_2-E_1)t} + e^{-\frac{i}{\hbar}(E_2-E_1)t}\right) dx =$$

$$\frac{1}{2} + \frac{4}{5a}\cos\frac{E_2-E_1}{\hbar}t\int_0^a \cos\frac{\pi x}{2a}\sin\frac{2\pi x}{2a}\,dx$$

$$P[t] = \frac{1}{2} + \frac{16}{15\pi}\cos\omega t$$




## Pricakovana vrednost koordinate

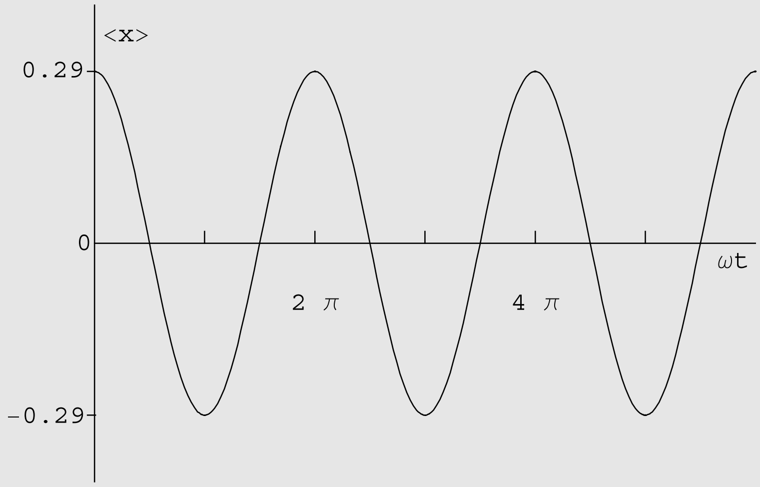
$$<x> = \int_{-a}^{a} \psi^* \, x \, \psi \, dx =$$

$$\int_{-a}^{a} \left( \frac{4}{5a} \cos^2 \frac{\pi x}{2a} + \frac{1}{5a} \sin^2 \frac{2\pi x}{2a} + \frac{2}{5a} \cos \frac{\pi x}{2a} \sin \frac{2\pi x}{2a} \cos \frac{(E_2 - E_1)\, t}{\hbar} \right)$$

$$x \, dx =$$

$$\frac{2}{5a} \cos \frac{(E_2 - E_1)\, t}{\hbar} \int_{-a}^{a} \cos \frac{\pi x}{2a} \sin \frac{2\pi x}{2a} \, x \, dx = \frac{128\, a}{45\, \pi^2} \cos\left(\frac{E_2 - E_1}{\hbar}\, t\right)$$

$$< x > = \frac{128\,a}{45\,\pi^2} \cos(\omega t)$$




# Pricakovana vrednost gibalne kolicine

## z integralom

odvod funcije y vstavimo v enacbo:

$$< \hat{p} > = \int -\psi^* \, i\hbar \frac{\partial}{\partial x} \psi \, dx =$$

$$\int_{-a}^{a} \Psi^* \, i\hbar \left( \frac{2\pi}{2\,a\sqrt{5\,a}} \sin \frac{\pi x}{2\,a} e^{-\frac{i}{\hbar} E_1 t} - \frac{\pi}{a\sqrt{5\,a}} \cos \frac{2\pi x}{2\,a} e^{-\frac{i}{\hbar} E_2 t} \right) dx$$

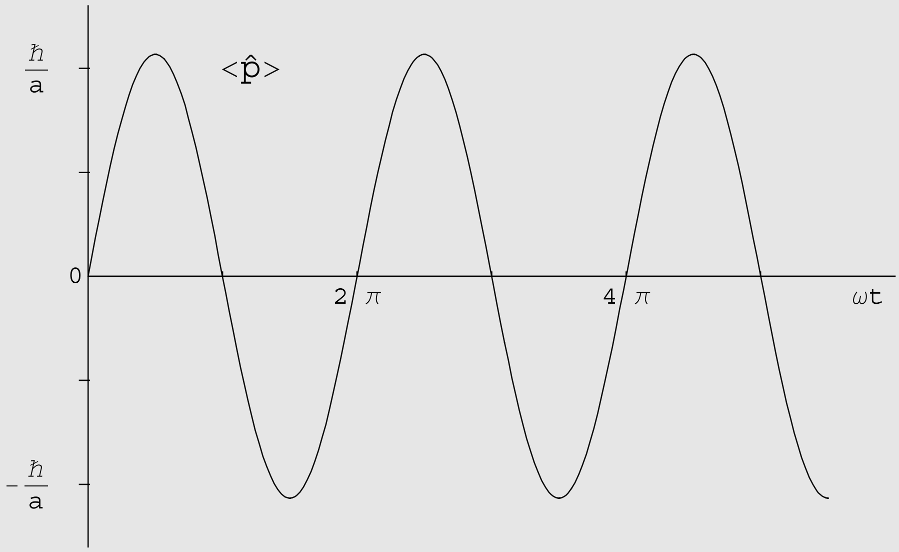
vzamemo samo sode clene produkta, ker je integral ostalih 0:

$$= \int_{-a}^{a} i\hbar \left( \frac{2\pi}{10\,a^2} \sin \frac{\pi x}{2\,a} \sin \frac{2\pi x}{2\,a} e^{-\frac{i}{\hbar}(E_1 - E_2)\,t} - \frac{2\pi}{5\,a^2} \cos \frac{2\pi x}{2\,a} \cos \frac{\pi x}{2\,a} e^{-\frac{i}{\hbar}(E_2 - E_1)\,t} \right) =$$

$$-\frac{\hbar}{5\,a} \sin\left( \frac{E_2 - E_1}{\hbar} t \right) \int_{-a}^{a} \frac{\pi}{a} \left( \sin \frac{\pi x}{2\,a} \sin \frac{\pi x}{a} - 2 \cos \frac{\pi x}{a} \cos \frac{\pi x}{2\,a} \right) dx$$

dobimo rezultat:

$$\langle \hat{p} \rangle = -\hbar \frac{16}{15\,a} \sin(\omega t)$$




## z odvodom koordinate:

$$\langle \hat{p} \rangle = m \frac{d \langle x \rangle}{dt}$$

$$\langle x \rangle = \frac{128\,a}{45\,\pi^2} \cos(\omega t)$$

pri izracunu bomo potrebovali lastne energije:

$$\omega = \frac{E_2 - E_1}{\hbar} = \frac{3}{2} \frac{\hbar\,\pi^2}{(2\,a)^2\,m} = \frac{3}{4} \frac{\hbar\,\pi^2}{a^2\,m}$$

$$\langle \hat{p} \rangle = m \frac{d}{dt} \left( \frac{128\,a}{45\,\pi^2} \cos(\omega t) \right) =$$
$$-m \frac{128\,a}{45\,\pi^2}\, \omega \sin(\omega t) = -m \frac{128\,a}{45\,\pi^2} \frac{3}{8} \frac{\hbar\,\pi^2}{a^2\,m} \sin(\omega t)$$

Vidimo, da je rezultat enak kot prej:

$$\langle \hat{p} \rangle = -\hbar \frac{16}{15\,a} \sin(\omega t)$$